MAE/MSE 502, Spring 2021 Homework #5

See the cover page of Homework #1 for the rules on collaboration. For all problems this homework, we expect a closed-form analytic solution without any unevaluated integrals. The solution must be written explicitly as a function of x and t (or explicitly as a function of x, y, and t for Problem 2).

Problem 1 (1 point)
For $u(x,t)$ defined on the domain of $-\infty < x < \infty$ and $t \geq 0$, solve the PDE

$$(1+t)\frac{\partial u}{\partial t} + t\,x\frac{\partial u}{\partial x} = t\,u$$

with the boundary condition

$$u(x,0) = e^{-x^2}\ .$$

Problem 2 (2 points)
For $u(x, y, t)$ defined on the domain of $-\infty < x < \infty$, $-\infty < y < \infty$, and $t \geq 0$, solve the PDE

$$(1+t)\frac{\partial u}{\partial t} + y\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = (1+t)u$$

with the boundary condition,

$$u(x,y,0) = e^{-(x^2+y^2)}\ .$$

Problem 3 (3 points)
For $u(x,t)$ defined on the domain of $-\infty < x < \infty$ and $t \geq 0$, solve the PDE

$$\frac{\partial u}{\partial t} + (0.5+u)\frac{\partial u}{\partial x} = -u$$

with the boundary condition,

$$u(x,0) = \begin{cases} 0\,, & if\ x < 0 \\ x^2, & if\ 0 \leq x \leq 1 \\ 1\,, & if\ x > 1 \end{cases}$$

Plot the solution, $u(x, t)$, as a function of x at $t = 0.4$ and 0.8. In addition, superimpose the initial state, $u(x, 0)$, as given by the boundary condition. Please collect all three curves in a single plot. The recommended range for plotting is $-1 \leq x \leq 3$.

Problem 4 (3 points)
For $u(x,t)$ defined on the domain of $-\infty < x < \infty$ and $t \geq 0$, solve the PDE

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = x$$

with the boundary condition,

$$u(x, 0) = 1$$

What is the steady solution (as $t \to \infty$) for the system? Plot the solution, $u(x, t)$, as a function of x at $t = 0.3$ and 1.0. In addition, superimpose the initial state, $u(x, 0)$, and the steady solution, $u_s(x)$. Please collect all four curves in a single plot. The recommended range for plotting is $-3 \leq x \leq 3$.

Problem 5 (3 points)
For $u(x,t)$ defined on the domain of $-\infty < x < \infty$ and $t \geq 0$, solve the PDE

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial t} + 2\frac{\partial u}{\partial x} = 0$$

with the boundary conditions,

(i) $u(x, 0) = 1$

(ii) $u_t(x, 0) = x$.

Hint: The differential operator, $\mathcal{L} \equiv \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)$, can be decomposed into $\mathcal{L} \equiv \mathcal{L}_1\mathcal{L}_2$ or $\mathcal{L} \equiv \mathcal{L}_2\mathcal{L}_1$, where $\mathcal{L}_1 \equiv \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)$ and $\mathcal{L}_2 \equiv \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)$. For this problem, you might find one of the decompositions more useful than the other.