Q2 By separation of variables, $u \sim G(x)H(t)$

$$\Rightarrow \quad e^t \frac{\dot{H}}{H} - 4\pi^2 = \boxed{\frac{G''}{G} = c \qquad G(0)=0,\ G(1)=0}$$

$$\Rightarrow \quad C_n = -(n\pi)^2 \quad n = 1, 2, 3, \ldots$$

$$G_n(x) = \sin(n\pi x), \ n = 1, 2, \ldots$$

From b.c. (iii), only $n = 1, 2$ matter

$$n = 1: \quad e^t \frac{\dot{H}_1}{H_1} - 4\pi^2 = -\pi^2$$

$$\Rightarrow \quad \frac{\dot{H}_1}{H_1} = 3\pi^2 e^{-t} \quad \Rightarrow \quad H_1(t) = \underbrace{H_1(0)}_{\text{ok to set to 1.}} e^{3\pi^2(1-e^{-t})}$$

$$n = 2: \quad \dot{H}_2 = 0 \quad \Rightarrow \quad H_2(t) = \text{const} \leftarrow \text{ok to set to 1.}$$

Full solution: $u(x,t) = a_1 G_1(x) H_1(t) + a_2 G_2(x) H_2(t)$

$$= a_1 \sin(\pi x)\, e^{3\pi^2(1-e^{-t})} + a_2 \sin(2\pi x)$$

From b.c. (iii), $a_1 = 1$, $a_2 = 1$.

$$\Rightarrow \quad u(x,t) = \sin(\pi x)\, e^{3\pi^2(1-e^{-t})} + \sin(2\pi x) \quad \#$$ — Full solution

Taking $t \to \infty$

$\Rightarrow$ steady solution is $u_s(x) = \sin(\pi x)\, e^{3\pi^2} + \sin(2\pi x)$ #

Q3 By sep. of var., $u(x,t) \sim G(x)H(t)$

$$\Rightarrow (1+2t)\frac{\dot{H}}{H} + 2t = \boxed{\frac{G''}{G} = c \quad G'(0)=0, \; G'(\pi)=0}$$

$\Rightarrow c = 0, \; -n^2, \; n = 1, 2, 3, \dots$

$G_0(x) = 1 \quad G_n(x) = \cos(nx),$
$n = 1, 2, 3, \dots$

From b.c. (iii), only $c=0$ and $c=-1$ matter.

$$c=0: \quad (1+2t)\frac{\dot{H}_0}{H_0} + 2t = 0$$

$$\Rightarrow \frac{\dot{H}_0}{H_0} = \frac{-2t}{1+2t} \Rightarrow H_0(t) = \underline{H_0(0)}\, e^{-t} \cdot \sqrt{1+2t}$$

$\rightarrow$ ok to set to 1.

~~$c=1$~~ $c=-1$: $(1+2t)\frac{\dot{H}_1}{H_1} + 2t = -1$

$$\Rightarrow \frac{\dot{H}_1}{H_1} = -1, \quad H_1(t) = \underline{H_1(0)}\, e^{-t}$$

$\rightarrow$ ok to set to 1.

Full solution:

$$u(x,t) = a_0 G_0(x) H_0(t) + a_1 G_1(x) H_1(t)$$
$$= a_0 e^{-t} \cdot \sqrt{1+2t} + a_1 \cos(x) e^{-t}$$

From b.c. (iii), $a_0 = 1$, $a_1 = 1$.

$$\Rightarrow u(x,t) = e^{-t} \cdot \sqrt{1+2t} + \cos(x) e^{-t}$$

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