Q2b By separation of variables, $u(x,y) \sim G(x)H(y)$:

$$-16\,\frac{G''}{G} = \boxed{\frac{\ddot{H}}{H} = c \qquad \dot{H}(0)=0 \quad \dot{H}(1)=0} \Rightarrow c = 0,\ -(n\pi)^2$$

$$n = 1, 2, 3, \ldots$$

From b.c. (i), (ii), only $c=0$ and $c=-(8\pi)^2$ matter.

$H_0(y) = 1$

$H_n(y) = \cos(n\pi x)$, $n=1,2,3,\ldots$

$$c=0 : \ G_0'' = 0 \ \Rightarrow \ G_0(x) = A_0 x + B_0$$

$$c = -(8\pi)^2 : \ -16\,\frac{G_8''}{G_8} = -(8\pi)^2 \ \Rightarrow \ G_8'' = (2\pi)^2 G_8$$

$$\Rightarrow \ G_8(x) = A_8 \cosh(2\pi x) + B_8 \sinh(2\pi x)$$

Full solution:

$$u(x,y) = (A_0 x + B_0) + \left[A_8 \cosh(2\pi x) + B_8 \sinh(2\pi x)\right]\cos(8\pi y)$$

$$\Rightarrow u_x(x,y) = A_0 + \left[2\pi A_8 \sinh(2\pi x) + 2\pi B_8 \cosh(2\pi x)\right]\cos(8\pi y)$$

b.c. (i) $\Rightarrow \ 1 = A_0 + 2\pi B_8 \cos(8\pi y) \ \Rightarrow \ A_0 = 1,\ B_8 = 0$

b.c. (ii) $\Rightarrow \ 1 + \cos(8\pi y) = A_0 + 2\pi A_8 \sinh(2\pi)\cos(8\pi y)$

$$\Rightarrow \ A_0 = 1, \ A_8 = 1/[2\pi \sinh(2\pi)].$$

B_0 remains undetermined (i.e., it can have an arbitrary value)

$$\Rightarrow \ u(x,y) = x + B_0 + \frac{\cosh(2\pi x)\cos(8\pi y)}{2\pi \sinh(2\pi)}$$

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arbitrary value

$\Rightarrow$ multiple solutions.

Q3 By sep. of var., $u(x,y) \sim G(x)H(y)$

$$\Rightarrow \quad -\left(\frac{G''}{G}+16\right) = \boxed{\frac{\ddot{H}}{H} = c \qquad \dot{H}(0)=0 \quad \dot{H}(\pi)=0}$$

$\Rightarrow$ $c = 0,\ -n^2$ $\quad n = 1, 2, 3, \ldots$

$H_0(y) = 1$, $H_n(y) = \cos(ny)$

From b.c. (i),(ii), only $c = 0,\ -4^2,\ -5^2$ matter.

$c = 0$: $-\left(\frac{G_0''}{G_0}+16\right) = 0$, $G_0'' = -16\,G_0$

$G_0(x) = A_0\cos(4x) + B_0\sin(4x)$

$c = -4^2$: $-\left(\frac{G_4''}{G_4}+16\right) = -16$, $G_4'' = 0 \Rightarrow G_4(x) = A_4 x + B_4$

$c = -5^2$: $-\left(\frac{G_5''}{G_5}+16\right) = -25$, $G_5'' = 9\,G_5$

$G_5(x) = A_5\cosh(3x) + B_5\sinh(3x)$

Full solution:
$$u(x,y) = A_0\cos(4x) + B_0\sin(4x) + (A_4 x + B_4)\cos(4y) + [A_5\cosh(3x) + B_5\sinh(3x)]\cos(5y)$$

b.c. (i) $\Rightarrow$ $0 = A_0 + B_4\cos(4y) + A_5\cos(5y) \Rightarrow A_0 = 0,\ B_4 = 0,\ A_5 = 0$

b.c. (ii) $\Rightarrow$ $1 + \cos(4y) + \cos(5y) = B_0\sin(4) + A_4\cos(4y) + B_5\sinh(3)\cos(5y)$

$\Rightarrow$ $B_0 = 1/\sin(4)$, $A_4 = 1$, $B_5 = 1/\sinh(3)$

Full solution:
$$u(x,y) = \frac{\sin(4x)}{\sin(4)} + x\cos(4y) + \frac{\sinh(3x)}{\sinh(3)}\cos(5y)$$

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Q4 By sep. of var, $u(x,y) \sim G(x)\,H(y)$

$$\Rightarrow \quad -\left[\frac{y^2\ddot{H} + 4y\dot{H} + 12H}{H}\right] = \boxed{\frac{G''}{G} = c \quad G(0)=0 \quad G(\pi)=0}$$

$$\Rightarrow \quad c_n = -n^2, \quad n = 1, 2, 3, \ldots$$

$$G_n(x) = \sin(nx)$$

From b.c. (iii), (iv), only $\underline{\underline{n=4}}$ matters.

$$n=4: \quad y^2\ddot{H}_4 + 4y\dot{H}_4 + 12H_4 = +16H_4$$

$$\Rightarrow \quad y^2\ddot{H}_4 + 4y\dot{H}_4 - 4H_4 = 0 \qquad \text{Let } \underline{H_4 \sim y^p}$$

$$\Rightarrow \quad p(p-1) + 4p - 4 = 0 \quad \Rightarrow \quad p^2 + 3p - 4 = 0$$

$$p = 1, \ -4$$

$$\Rightarrow \quad H_4(y) = A_4 y + \frac{B_4}{y^4}$$

Full solution: $u(x,y) = \left(A_4 y + \frac{B_4}{y^4}\right)\sin(4x)$

$$\left.\begin{array}{ll} \text{b.c. (iii)} \Rightarrow & 0.5A_4 + 16B_4 = 17 \\ \text{b.c. (iv)} \Rightarrow & A_4 + B_4 = 3 \end{array}\right\} \text{ solve} \Rightarrow A_4 = 2, \ B_4 = 1$$

Full solution:

$$u(x,y) = \left(2y + \frac{1}{y^4}\right)\sin(4x)$$

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