MAE/MSE 502, Fall 2022 Homework #3

Please follow the rules on collaboration as given in Homework #1. A statement of collaboration is required.

Problem 1 (2.5 points)
For $u(x, t)$ defined on the domain of $0 \le x \le 100$ and $t \ge 0$, consider the 1-D Wave equation,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \ ,$$

with the boundary conditions,

(i) $u(0, t) = 0$, (ii) $u(100, t) = 0$, (iii) $u(x, 0) = P(x)$, (iv) $u_t(x, 0) = 0$.

(a) Solve the system with $P(x)$ given as

$$P(x) = \begin{cases} x/25 \, , & if\ 0 \le x \le 25 \\ (100 - x)/75 \, , & if\ 25 < x \le 100 \end{cases}$$

and plot the solution as a function of x at t = 0, 30, 50, 70, 100, 180. Please collect all 6 curves in one plot. (See remarks below HW1-Prob1 on proper truncation of infinite series.)

(b) Solve the system with $P(x)$ given as (this emulates a "wave packet")

$$P(x) = \sum_{n=30}^{50} \exp\left[-\left(\frac{n-40}{4}\right)^2\right] \sin\left(\frac{n\pi x}{100}\right) ,$$

and plot the solution as a function of x at t = 0, 50, 100, and 125. For this part, it is recommended that the four curves be plotted separately, for example arranged in 4 panels using "subplot" in Matlab.

Problem 2 (3 points)
For $u(x,t)$ defined on the domain of $0 \le x \le 2\pi$ and $t \ge 0$, consider the PDE (in which U, K, and B are constants)

$$\frac{\partial u}{\partial t} = U \frac{\partial u}{\partial x} + K \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^3 u}{\partial x^3}$$

with periodic boundary conditions in the x-direction. The boundary condition at $t = 0$ is given as

$$u(x, 0) = \frac{[1 - \cos(x)]^8}{256} \ .$$

Solve the PDE and plot the solution $u(x, t)$ at $t = 0.2$ for the four cases: **(i)** ($U = 10, K = 0, B = 0$), **(ii)** ($U = 0, K = 1.5, B = 0$), **(iii)** ($U = 10, K = 1.5, B = 0$), and **(iv)** ($U = 0, K = 0, B = 0.15$), Also, plot the initial state $u(x, 0)$ (which is the same for all four cases). Please collect all five curves in one plot. [The solution can be expressed as an infinite series. To compute the value of u, the series needs to be truncated to a finite number of terms (see remarks below HW1-Prob1). Matlab can be used to evaluate the expansion coefficients by numerical integration.]

For Problem 3-5, we expect a closed-form solution which consists of only a finite number of terms and no unevaluated integrals. The solution should be expressed in real numbers and functions. Expect a deduction if an imaginary number "i" (=$\sqrt{-1}$) is left in the solution.

Problem 3 (2.5 points)
For $u(x, t)$ defined on the domain of $0 \le x \le 2\pi$ and $t \ge 0$, solve the PDE,

$$\frac{\partial u}{\partial t} = t\frac{\partial^5 u}{\partial x^5} + t\frac{\partial^3 u}{\partial x^3} + t^2\frac{\partial^2 u}{\partial x^2} + 4\,t^2\,u\ ,$$

with periodic boundary conditions in the x-direction. The boundary condition in the t-direction is given as

$u(x, 0) = \sin(x) + \cos(x) + \cos(2x)$.

Problem 4 (2.5 points)
For $u(x, t)$ defined on the domain of $0 \le x \le 2\pi$ and $t \ge 0$, solve the PDE,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^4 u}{\partial x^4} + 4\frac{\partial^2 u}{\partial x^2} + 4\,u\ ,$$

with periodic boundary conditions in the x-direction. The boundary conditions in the t-direction are given as

(i) $u(x, 0) = \sin(x)$
(ii) $u_t(x, 0) = \cos(2x)$.

Problem 5 (2.5 points)
For $u(x,t)$ defined on the domain of $0 \le x \le 2\pi$ and $t \ge 0$, solve the PDE

$$\frac{\partial^2 u}{\partial t^2} - 4\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x\,\partial t} - 3\frac{\partial u}{\partial t} - 4\,u = 0\,,$$

with periodic boundary conditions in the x-direction, and the boundary conditions at $t = 0$ given as

(i) $u(x, 0) = 3 + \cos(x)$
(ii) $u_t(x, 0) = 2 + \cos(x)$.