Q3

By Fourier Series expansion:

$$u(x,t) = \sum_n C_n(t) e^{inx}$$

$$\Rightarrow \dot{C}_n = \left[i(n^5 - n^3)t + (4-n^2)t^2\right] C_n$$

From b.c., only $n=1,2$ need to be considered.

From b.c.:

$$u(x,0) = \left[\left(\frac{1}{2i} + \frac{1}{2}\right)e^{ix} + \frac{1}{2}e^{i2x}\right] + \text{c.c.}$$

$$\Rightarrow C_1(0) = \frac{1}{2i} + \frac{1}{2} = \frac{1}{2} - \frac{i}{2}$$

$$C_2(0) = \frac{1}{2}$$

all other $C_n(0) = 0$

$n=1$: $\quad \dot{C}_1 = 3t^2 C_1 \Rightarrow C_1(t) = C_1(0) e^{t^3}$

$$\Rightarrow C_1(t) = \left(\frac{1}{2} - \frac{i}{2}\right) e^{t^3}$$

$n=2$: $\quad \dot{C}_2 = i24t\, C_2 \Rightarrow C_2(t) = C_2(0) e^{i12t^2}$

$$\Rightarrow C_2(t) = \frac{1}{2} e^{i \cdot 12t^2}$$

Full solution:

$$u(x,t) = \left[C_1(t) e^{ix} + C_2(t) e^{i2x}\right] + \text{c.c.}$$

$$= \left[e^{t^3}\left(\frac{1}{2} - \frac{i}{2}\right) e^{ix} + \frac{1}{2} e^{i(2x + 12t^2)}\right] + \text{c.c.}$$

$$= e^{t^3}(\cos(x) + \sin(x)) + \cos(2x + 12t^2)$$

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Note:

$$\left(\frac{1}{2} - \frac{i}{2}\right) e^{ix} = \left(\frac{1}{2} - \frac{i}{2}\right)(\cos(x) + i\sin(x))$$

$$= \underbrace{\left[\frac{1}{2}\cos(x) + \frac{1}{2}\sin(x)\right]}_{\text{real part to keep and double}} + i\left[\frac{1}{2}\sin(x) - \frac{1}{2}\cos(x)\right]$$

Q4

Fourier series expansion:

$$u(x,t) = \sum_n C_n(t) e^{inx}$$

$$\Rightarrow \ddot{C}_n = (n^4 - 4n^2 + 4) C_n$$

From b.c. (i) and (ii), consider only $n=1,2$

$n=1$: $\ddot{C}_1 = C_1$

$\Rightarrow C_1(t) = A_1 \cosh(t) + B_1 \sinh(t)$

$\rightarrow \dot{C}_1(t) = A_1 \sinh(t) + B_1 \cosh(t)$

with $C_1(0) = \frac{1}{2i}$, $\dot{C}_1(0) = 0$

$\Rightarrow A_1 = \frac{1}{2i}$, $B_1 = 0$

$\Rightarrow C_1(t) = \frac{1}{2i}\cosh(t)$.

From b.c. (i):

$u(x,0) = \sin(x)$

$= \frac{1}{2i} e^{ix} + c.c.$

$\Rightarrow C_1(0) = \frac{1}{2i}$

$C_n(0) = 0$ for $n \neq 1$

From b.c. (ii):

$u_t(x,0) = \cos(2x)$

$= \frac{1}{2} e^{i2x} + c.c.$

$\Rightarrow \dot{C}_2(0) = \frac{1}{2}$

$\dot{C}_n(0) = 0$ for $n \neq 2$

$n=2$: $\ddot{C}_2 = 4C_2$

$\Rightarrow C_2(t) = A_2 \cosh(2t) + B_2 \sinh(2t)$

$\rightarrow \dot{C}_2(t) = 2A_2 \sinh(2t) + 2B_2 \cosh(2t)$

with $C_2(0) = 0$, $\dot{C}_2(0) = \frac{1}{2}$ $\Longrightarrow$ $A_2 = 0$, $B_2 = \frac{1}{4}$

$\Rightarrow C_2(t) = \frac{1}{4}\sinh(2t)$.

Full solution: $u(x,t) = [C_1(t) e^{ix} + C_2(t) e^{i2x}] + c.c.$

$$= \left[\frac{1}{2i}\cosh(t) e^{ix} + \frac{1}{4}\sinh(2t) e^{i2x}\right] + c.c.$$

$$= \sin(x)\cosh(t) + \frac{1}{2}\cos(2x)\sinh(2t)$$

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Q5

Fourier series expansion: $u(x,t) = \sum_n C_n(t) e^{inx}$

$$\Rightarrow \ddot{C}_n - (3+in)\dot{C}_n + (4n^2-4)C_n = 0$$

From b.c. (i) & (ii), only $n=0,1$ matter.

From b.c. (i)

$$u(x,0) = 3 + \cos(x) = 3 + \left(\tfrac{1}{2}e^{ix} + c.c.\right)$$

$$\Rightarrow C_0(0) = 3, \quad C_1(0) = \tfrac{1}{2}$$

From b.c. (ii)

$$u_t(x,0) = 2 + \cos(x) = 2 + \left(\tfrac{1}{2}e^{ix} + c.c.\right)$$

$$\Rightarrow \underline{\dot{C}_0(0) = 2}, \ \underline{\dot{C}_1(0) = \tfrac{1}{2}}$$

$\underline{\underline{n=0}}$ $\Rightarrow \ddot{C}_0 - 3\dot{C}_0 - 4C_0 = 0$

Let $C_0(t) \sim e^{\alpha t} \Rightarrow \alpha^2 - 3\alpha - 4 = 0$, $\underline{\alpha = 4, -1}$

$$\Rightarrow C_0(t) = A_0 e^{4t} + B_0 e^{-t}$$

$$\rightarrow \dot{C}_0(t) = 4A_0 e^{4t} - B_0 e^{-t}$$

Using the b.c. for $n=0$:

$$\left.\begin{aligned} C_0(0) &= A_0 + B_0 = 3 \\ \dot{C}_0(0) &= 4A_0 - B_0 = 2 \end{aligned}\right\} \Rightarrow A_0 = 1,\ B_0 = 2$$

$$\Rightarrow C_0(t) = e^{4t} + 2e^{-t}$$

$\underline{\underline{n=1}}$ $\Rightarrow \ddot{C}_1 = (3+i)\dot{C}_1$ Let $D(t) \equiv \dot{C}_1(t)$, then $\dot{D} = (3+i)D$

$$\Rightarrow D(t) = D(0)e^{(3+i)t} = \tfrac{1}{2}e^{(3+i)t}$$

Also, $D(0) = \dot{C}_1(0) = \tfrac{1}{2}$

$$\Rightarrow \dot{C}_1 = \tfrac{1}{2}e^{(3+i)t} \Rightarrow C_1(t) = \underbrace{C_1(0)}_{=1/2} + \frac{1}{2(3+i)}\left[e^{(3+i)t} - 1\right]$$

Full solution:

$$u(x,t) = C_0(t) + \left[C_1(t)e^{ix} + c.c.\right]$$

$$= e^{4t} + 2e^{-t} + \left\{\underbrace{\left(\frac{1}{2} + \frac{1}{2(3+i)}\left[e^{(3+i)t} - 1\right]\right)e^{ix}}_{\star} + c.c.\right\}$$

$\star$ see next page for detail

$$= e^{4t} + 2e^{-t} +$$

$$+ \cos(x) + \frac{1}{10}\left[e^{3t}\left(3\cos(x+t) + \sin(x+t)\right) - \left(3\cos(x) + \sin(x)\right)\right]$$

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Note for Q5 :

$$\star = \left(\frac{1}{2} + \frac{1}{2(3+i)}\left[e^{(3+i)t} - 1\right]\right) e^{ix}$$

$$= \underbrace{\frac{1}{2} e^{ix}}_{1} + \frac{1}{20} e^{3t} \cdot \underbrace{(3-i)e^{i(x+t)}}_{2} - \frac{1}{20} \underbrace{(3-i)e^{ix}}_{3}$$

$$\frac{1}{3+i} = \frac{3-i}{(3+i)(3-i)} = \frac{3-i}{10}$$

$$1 = \underbrace{\cos(x)}_{\text{real part to keep}} + i\sin(x)$$

$$2 = (3-i)\left(\cos(x+t) + i\sin(x+t)\right)$$

$$= \underbrace{\left[3\cos(x+t) + \sin(x+t)\right]}_{\text{real part to keep}} + i\left[3\sin(x+t) - \cos(x+t)\right]$$

$$3 = (3-i)\left(\cos(x) + i\sin(x)\right)$$

$$= \underbrace{\left[3\cos(x) + \sin(x)\right]}_{\text{real part to keep}} + i\left[3\sin(x) - \cos(x)\right]$$

putting together, we extract $2 \times$ (real part of $\star$) as:

$$2 \cdot Re(\star) = \cos(x) + \frac{1}{10}\left[e^{3t}\left(3\cos(x+t) + \sin(x+t)\right) - \left(3\cos(x) + \sin(x)\right)\right].$$