Q2

The building blocks in x-direction are $\{1, \cos(nx)\}$, $n = 1, 2, 3, \ldots$

$\Rightarrow$ Let $u(x,t) = a_0(t) + \sum_{n=1}^{\infty} a_n(t)\cos(nx)$ → From b.c. (iii),

$$\underbrace{t + e^{-t}\cos(x)} = Q(x,t) = \underbrace{q_0(t) + \sum_{n=1}^{\infty} q_n(t)\cos(nx)}$$

$a_0(0) = 0$

$a_1(0) = 0$

compare $\Rightarrow q_0(t) = t \quad q_1(t) = e^{-t}$

The ODE for a_n is:

$$\dot{a}_n = (n^2 - 1)\, a_n + q_n$$

Only $\underline{n = 0, 1}$ matter.

$$n=0 \to \quad \dot{a}_0 = -a_0 + t \quad \Rightarrow a_0(t) = \overset{=0}{\widetilde{a_0(0)}}\, e^{-t} + \int_0^t \hat{t}\, e^{-(t-\hat{t})}\, d\hat{t}$$

$$= \left(1 - e^{t} + t e^{t}\right) e^{-t}$$

$$= e^{-t} - 1 + t$$

$$n=1 \Rightarrow \quad \dot{a}_1 = e^{-t} \quad \Rightarrow a_1(t) = \underbrace{a_1(0)}_{=0} + \int_0^t e^{-t}\, dt = 1 - e^{-t}$$

Full solution:

$$u(x,t) = a_0(t) + a_1(t)\cos(x)$$

$$= \left(e^{-t} - 1 + t\right) + \left(1 - e^{-t}\right)\cos(x)$$

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Q3

The steady solution $u_s(x)$ satisfies

$$u_s'' = -2 - \pi^2 \cos(\pi x), \quad \underbrace{u_s(0)=1}_{①}, \quad \underbrace{u_s(1)=-1}_{②}$$

$$\Rightarrow u_s(x) = -x^2 + \cos(\pi x) + Ax + B$$

From b.c. ① & ② $\Rightarrow A = 1,\ B = 0$

$$\Rightarrow u_s(x) = \cos(\pi x) + x - x^2$$

Let $\hat{u}(x,t) \equiv u(x,t) - u_s(x)$,

then the original system becomes:

$$(\star)\begin{cases} \dfrac{\partial \hat{u}}{\partial t} = \dfrac{\partial^2 \hat{u}}{\partial x^2} \\ \hat{u}(0,t) = 0 \\ \hat{u}(1,t) = 0 \\ \hat{u}(x,0) = \sin(\pi x) \end{cases}$$

The solution of $(\star)$ is $\hat{u}(x,t) = \sin(\pi x)\, e^{-\pi^2 t}$

$\Rightarrow$ Full solution:

$$\begin{aligned} u(x,t) &= \hat{u}(x,t) + u_s(x) \\ &= \sin(\pi x)\, e^{-\pi^2 t} + \cos(\pi x) + x - x^2 \end{aligned}$$

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Q4

By Fourier series expansion:

$$u(x,t) = \sum_{n=-\infty}^{\infty} C_n(t) e^{inx} \Longrightarrow$$

From b.c. (i): $C_0(0) = 3$, $C_1(0) = 0$

$$e^{-t} + \cos(x) = Q(x,t) = \sum_{n=-\infty}^{\infty} q_n(t) e^{inx}$$

From b.c. (ii): $\dot{C}_0(0) = 1$, $\dot{C}_1(0) = 0$

comparison $\Rightarrow q_0(t) = e^{-t}$, $q_1(t) = \frac{1}{2}$

The ODE for $C_n(t)$ is:

$$\ddot{C}_n + \dot{C}_n - n^2 \dot{C}_n + n^2 C_n = q_n$$

Only $n = 0, 1$ (and -1) matter.

$n=0 \rightarrow \ddot{C}_0 + \dot{C}_0 = e^{-t}$ Let $D(t) \equiv \dot{C}_0(t)$, $\Rightarrow D(0) = \dot{C}_0(0) = 1$

then $\dot{D} = -D + e^{-t} \Rightarrow D(t) = D(0) e^{-t} + \int_0^t e^{-\hat{t}} e^{-(t-\hat{t})} d\hat{t}$

$= (1+t) e^{-t}$

$\dot{C}_0 = D = (1+t) e^{-t} \Rightarrow C_0(t) = \underbrace{C_0(0)}_{3} + \int_0^t (1+t) e^{-t} dt$

$= 5 - 2e^{-t} - te^{-t}$

$n=1 \rightarrow \ddot{C}_1 = -C_1 + \frac{1}{2}$

Let $E \equiv C_1 - \frac{1}{2} \Rightarrow \ddot{E} = -E \Rightarrow E(t) = A\cos(t) + B\sin(t)$

$\Rightarrow C_1 = \frac{1}{2} + A\cos(t) + B\sin(t)$ $C_1(0) = 0 \Rightarrow \frac{1}{2} + A = 0$

$\rightarrow \dot{C}_1(t) = -A\sin(t) + B\cos(t)$ $A = -\frac{1}{2}$

$\dot{C}_1(0) = 0 \Rightarrow B = 0$

$\Rightarrow C_1(t) = \frac{1}{2}(1 - \cos(t)$

Full solution: $u(x,t) = C_0(t) + (C_1(t) e^{ix} + c.c.)$

$= C_0(t) + \{[\frac{1}{2}(1-\cos(t)] e^{ix} + c.c.\}$

$= (5 - 2e^{-t} - te^{-t}) + [1 - \cos(t)] \cos(x)$

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