Q1

PDE can be rearranged as

$$\frac{\partial u}{\partial t} - 2x \frac{\partial u}{\partial x} = x u e^{2t} + 1$$

by MOC: $\frac{dx}{dt} = -2x \Rightarrow x(t) = x(0) e^{-2t} \Rightarrow x(0) = x(t) e^{2t}$

$$\frac{du}{dt} = x u e^{2t} + 1$$
$$= x(0) e^{-2t} \cdot u e^{2t} + 1$$

$$\Rightarrow \frac{du}{dt} = x(0) u + 1 \quad \text{---} (*)$$

$$u(t) = u(0) e^{x(0)t} + \int_0^t 1 \cdot e^{x(0)(t-\hat{t})} d\hat{t}$$

$$= u(0) e^{x(0)t} + e^{x(0)t} \cdot \frac{1}{x(0)} [1 - e^{-x(0)t}]$$

From b.c. $u(0) = 1$ $\Rightarrow$

$$= e^{x(0)t} + \frac{1}{x(0)} [e^{x(0)t} - 1]$$

$$= e^{x(t)e^{2t} \cdot t} + \frac{1}{x(t)e^{2t}} [e^{x(t)e^{2t} \cdot t} - 1]$$

$$\Rightarrow u(x,t) = e^{xe^{2t} \cdot t} + \frac{1}{xe^{2t}} [e^{xe^{2t} \cdot t} - 1] \quad \text{---} (\maltese)$$

Note: In $(*)$, if $x(0) = 0$ (which implies $x(t) = 0$),

then $\frac{du}{dt} = 1 \Rightarrow u(t) = u(0) + t$

$\Rightarrow u(t) = 1 + t$

So, the solution is $\underline{u(x,t) = 1 + t, \text{ if } x = 0}$

This resolves the apparent singularity in the expression of $(\maltese)$ (which should be used for computation only for $x \neq 0$)

Q2

By MOC, $$\frac{dx}{dt} = 2u \quad \text{— ①}$$

$$\frac{du}{dt} = 2x + e^{2t} \quad \text{— ②}$$

Let $v(t) \equiv x(t) + u(t)$, then, adding ① & ②,

$$\frac{dv}{dt} = 2v + e^{2t} \Rightarrow v(t) = v(0)e^{2t} + \int_0^t e^{2\hat{t}} e^{2(t-\hat{t})} d\hat{t}$$

$$= v(0)e^{2t} + t e^{2t}$$

$$= t e^{2t}$$

$v(0) = x(0) + u(0)$ from b.c.

$= x(0) - x(0)$

$= 0$

So, $u(x) + x(t) = t e^{2t}$

Full solution is

$$u(x,t) = -x + t e^{2t}$$

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Q3

By MOC: $\dfrac{dx}{dt} = -u = -(u(0)+t) \Rightarrow x(t) = x(0) - u(0)t - \dfrac{t^2}{2}$

$\dfrac{du}{dt} = 1 \Rightarrow u(t) = u(0)+t$

Use shorthand: $x(t) \to x$, $x(0) \to x_0$

$$\begin{cases} x = x_0 - u(0)t - \dfrac{t^2}{2} & \text{--- (1)} \\ u(x,t) = u(0)+t & \text{--- (2)} \end{cases}$$

If $x_0 < 1 \Rightarrow u(0) = 1 \Rightarrow \boxed{u(x,t) = 1+t}$

$\Downarrow$

$x = x_0 - t - \dfrac{t^2}{2}$

$\boxed{x < 1 - t - \dfrac{t^2}{2}}$

If $x_0 \geq 1 \Rightarrow u(0) = \dfrac{1}{x_0} \Rightarrow u(x,t) = \dfrac{1}{x_0} + t$

$\Downarrow$

$x = x_0 - \dfrac{t}{x_0} - \dfrac{t^2}{2}$

$\Downarrow$

$x_0 - (x + \frac{t^2}{2})x_0 - t = 0$

express x_0 in (x,t)

$\boxed{x \geq 1 - t - \dfrac{t^2}{2}}$

$$x_0 = \frac{(x+\frac{t^2}{2}) \pm \sqrt{(x+\frac{t^2}{2})^2 + 4t}}{2}$$

only "+" root is valid

(the "−" root violates $x_0 \geq 1$)

Full solution:

$$u(x,t) = \begin{cases} 1+t, & \text{if } x < 1 - t - \dfrac{t^2}{2} \\[2ex] \dfrac{2}{(x+\frac{t^2}{2}) + \sqrt{(x+\frac{t^2}{2})^2 + 4t}} + t, & \text{if } x \geq 1 - t - \dfrac{t^2}{2} \end{cases}$$

Q4

Write the PDE as

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\underbrace{\left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\right)}_{w} = 0.5\,x$$

$$w(x,t) = u_t(x,t) + u_x(x,t)$$
$$\Rightarrow w(x,0) = u_t(x,0) + u_x(x,0) = 1$$

$$\Rightarrow \begin{cases} \dfrac{\partial w}{\partial t} - \dfrac{\partial w}{\partial x} = 0.5\,x & \text{— ①} \\[2ex] \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} = w & \text{— ②} \end{cases}$$

* Solve ① with b.c. : $w(x,0) = 1$ $\quad (w(0) = 1)$

By MOC, $\dfrac{dx}{dt} = -1 \Rightarrow x(t) = x(0) - t \Rightarrow x(0) = x(t) + t$

$$\frac{dw}{dt} = 0.5x = 0.5x(0) - 0.5t$$
$$\Rightarrow w(t) = w(0) + 0.5\,x(0)\cdot t - 0.25t^2$$
$$\Rightarrow w(x,t) = 1 + 0.5\,(x+t)\cdot t - 0.25t^2 = 1 + 0.5xt + 0.25t^2$$

* Solve ② with b.c.: $u(x,0) = x$ $\quad (u(0) = x(0))$

By MOC $\dfrac{dx}{dt} = 1 \Rightarrow x(t) = x(0) + t$
$\Rightarrow x(0) = x(t) - t$

$$\frac{du}{dt} = w = 1 + 0.5xt + 0.25t^2 = 1 + 0.5x(0)t + 0.75t^2$$
$$\Rightarrow u(t) = t + 0.25\,x(0)t^2 + 0.25t^3 + u(0)$$
$$\Rightarrow u(x,t) = t + 0.25(x-t)t^2 + 0.25t^3 + (x-t)$$
$$= x + 0.25\,xt^2$$

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