Q2

By sep. of var. : $u(x,t) \sim G(x)H(t)$

$\Rightarrow \quad G\dot{H} = (1+t)\,G''H + \pi^2(1+t)\,GH$

From b.c. (i), (ii)

$\Rightarrow \quad \dfrac{1}{1+t}\dfrac{\dot{H}}{H} - \pi^2 = \boxed{\dfrac{G''}{G} = c \qquad \begin{matrix} G(0)=0 \\ G(1)=0 \end{matrix}} \rightarrow c_n = -(n\pi)^2, \quad n = 1, 2, 3, \ldots$

$G_n(x) = \sin(n\pi x)$

observing b.c. (iii), only $n = 1, 2$ matter

OK to set to 1

$n=1 \Rightarrow \dfrac{1}{1+t}\dfrac{\dot{H}_1}{H_1} - \pi^2 = -\pi^2 \Rightarrow \dot{H}_1 = 0 \qquad H_1(t) = \text{const.}$

$n=2 \Rightarrow \dfrac{1}{1+t}\dfrac{\dot{H}_2}{H_2} - \pi^2 = -4\pi^2 \Rightarrow \dfrac{\dot{H}_2}{H_2} = -3\pi^2(1+t)$

$\Rightarrow \quad H_2(t) = \underbrace{H_2(0)}_{\text{OK to set to 1}}\, e^{-3\pi^2\left(t + \frac{t^2}{2}\right)}$

Full solution:

$$u(x,t) = a_1\, G_1(x)H_1(t) + a_2\, G_2(x)H_2(t)$$
$$= a_1 \sin(\pi x) + a_2 \sin(2\pi x)\, e^{-3\pi^2\left(t+\frac{t^2}{2}\right)}$$

From b.c. (iii), by visual inspection, $a_1 = 1$, $a_2 = 1$

$$\Rightarrow \quad u(x,t) = \sin(\pi x) + \sin(2\pi x)\, e^{-3\pi^2\left(t+\frac{t^2}{2}\right)}$$

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steady solution $u_s(x) = u(x, t\to\infty) = \underline{\sin(\pi x)}$

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Q3

By sep. of var.: $u(x,t) \sim G(x)H(t)$

$$\Rightarrow (1+t)\,G\dot{H} = G''H + 4GH$$

$$\Rightarrow (1+t)\frac{\dot{H}}{H} - 4 = \boxed{\frac{G''}{G} = c \qquad G'(0)=0,\ G'(\pi)=0} \rightarrow c = 0,\ -n^2,\ n=1,2,3,\ldots$$

From b.c. (i), (ii)

$G_0(x) = 1$

$G_n(x) = \cos(nx),\ n \neq 0$

observing b.c. (iii), only $n = 0, 1, 2$ matter

$$n=0 \Rightarrow \frac{\dot{H}_0}{H_0} = \frac{4}{1+t} \Rightarrow H_0(t) = \underbrace{H_0(0)}_{\text{set to } 1}(1+t)^4$$

$$n=1 \Rightarrow \frac{\dot{H}_1}{H_1} = \frac{3}{1+t} \Rightarrow H_1(t) = \underbrace{H_1(0)}_{\text{set to } 1}(1+t)^3$$

$$n=2 \Rightarrow \frac{\dot{H}_2}{H_2} = 0 \Rightarrow H_2(t) = \text{const.}$$

set to 1

Full solution:

$$u(x,t) = a_0 G_0(x)H_0(t) + a_1 G_1(x)H_1(t) + a_2 G_2(x)H_2(t)$$
$$= a_0(1+t)^4 + a_1\cos(x)(1+t)^3 + a_2\cos(2x)$$

From b.c. (iii), by visual inspection, $a_0 = 1$, $a_1 = 1$, $a_2 = 1$.

$$\Rightarrow u(x,t) = (1+t)^4 + \cos(x)(1+t)^3 + \cos(2x)$$

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