MAE/MSE 502, Spring 2022 Homework #2

Please follow the rules on collaboration as given in Homework #1. Your work should include a statement of collaboration.

Problem 1 (3 points)
For $u(x,y)$ defined on the square domain of $0 \le x \le 1$ and $0 \le y \le 1$, solve the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\,,$$

with the boundary conditions,

(i) $u(0, y) = 0$ (ii) $u(1, y) = 1.5\sqrt{y}\sin(\pi y)$ (iii) $u(x, 0) = 0$ (iv) $u(x, 1) = 2.5\,(x - x^3)$.

Make a contour plot of the solution, $u(x, y)$. See Additional Note for an example of using Matlab to make a contour plot. The recommended contour interval for the plot is 0.1. For this problem, we expect the solution to be expressed as an infinite series. Please see the remark below HW1-Q1 on how to truncate the series and numerically compute the expansion coefficients.

Problem 2 (4 points)
(a) For $u(x, y)$ defined on the square domain of $0 \le x \le 1$ and $0 \le y \le 1$, solve the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 4\pi^2 u = 0$$

with the boundary conditions (u_x denotes $\partial u/\partial x$),

(i) $u_x(0, y) = 3\sin(2\pi y)$ (ii) $u(1, y) = 5\sin(2\pi y)$ (iii) $u(x, 0) = 0$ (iv) $u(x, 1) = 0$

Please also explicitly state whether the system has a unique solution, multiple solutions, or no solution.

(b) Repeat (a) but with the second boundary condition changed to

(ii) $u_x(1, y) = 3\sin(2\pi y)$

(Note that the b.c. is now imposed on u_x.) The other boundary conditions remain the same as Part (a).

For both (a) and (b), if a solution or solutions exist, we expect them to be expressed in a closed form with only a finite number of terms and without any unevaluated integrals. There will be a deduction otherwise.

Problem 3 (3 points)
For $u(x,y)$ defined on the rectangular domain of $0 \le x \le \pi$ and $1 \le y \le 2$, solve the PDE

$$\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} - 2y\frac{\partial u}{\partial y} + 5u = 0$$

with the boundary conditions (u_x denotes $\partial u/\partial x$),

(i) $u_x(0, y) = 0$ (ii) $u_x(\pi, y) = 0$ (iii) $u(x, 1) = 3\cos(3x)$ (iv) $u(x, 2) = 17\cos(3x)$

For this problem, we expect a closed-form solution that consists of only a finite number of terms and without any unevaluated integrals. There will be a deduction if the requirement is not satisfied.

Additional Note: Using Matlab to make a contour plot

The following Matlab code makes a contour plot for $u(x,y) = \sin(2\pi x) + x\cos(3\pi y)$ over the domain of $0 \le x \le 1$ and $0 \le y \le 1$, using the contour interval of 0.2 over the range of $-2 \le u \le 2$. Selected contours with $u = -1.6+0.4p$, $p = 1, 2, 3, \ldots$, are labeled. The contour plot is shown in the next page.
It is essential to provide the coordinates of the grid (x2d and y2d in this example) as the input for the contour function. Without this piece of information, Matlab would not know the grid spacing and the correct directions of x and y. Be aware that the setting used in this example might not be what is recommended for the actual contour plot in Problem 1. For that problem, please adjust the contour interval as needed.

```
x = [0:0.01:1];y = [0:0.01:1];
for m = 1:length(x)
    for n = 1:length(y)
        u(m,n) = sin(2*pi*x(m))+x(m)*cos(3*pi*y(n));
        x2d(m,n) = x(m); y2d(m,n) = y(n);
    end
end
[C h] = contour(x2d,y2d,u,[-2:0.2:2],'LineWidth',2)
clabel(C,h,[-1.6:0.4:1.6]); xlabel('x'); ylabel('y')
```