Q2

By separation of variables: $u \sim G(x)H(y)$,

From b.c. (iii), (iv)

$$\Rightarrow \quad -\left(\frac{G''}{G} + 4\pi^2\right) = \frac{\ddot{H}}{H} = c \qquad H(0) = 0, \quad H(1) = 0$$

$$\Rightarrow \quad c_n = -(n\pi)^2, \quad n = 1, 2, 3, \ldots$$

$$H_n(y) = \sin(n\pi y)$$

Observing b.c. (i), (ii), only $n=2$ matters

For $n=2$, $\quad \dfrac{G_2''}{G_2} = 4\pi^2 - 4\pi^2 = 0 \quad \Rightarrow \quad G_2(x) = A_2 x + B_2$

Full solution: $\quad u(x, y) = (A_2 x + B_2)\sin(2\pi y)$ ——— $(*)$

(a) From $(*)$ $\Rightarrow$ $u_x(x, y) = A_2 \sin(2\pi y)$

b.c. (i) $\rightarrow$ $A_2 = 3$, b.c. (ii) $\rightarrow$ $(3 \cdot 1 + B_2) = 5 \Rightarrow B_2 = 2$

Full solution is $u(x, y) = (3x + 2)\sin(2\pi y)$, unique.

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(b) b.c. (i) $\rightarrow$ $A_2 = 3$, b.c. (ii) $\rightarrow$ $A_2 = 3$

B_2 remains undetermined (it can be of any value)

Full solution is $u(x, y) = (3x + B_2)\sin(2\pi y)$,

infinite many solutions.

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Q3

By sep. of var., $u \sim G(x)H(y)$

$$\Rightarrow -\left(\frac{y^2\ddot{H} - 2y\dot{H} + 5H}{H}\right) = \boxed{\frac{G''}{G} = C \quad \begin{array}{l} G'(0)=0 \\ G'(\pi)=0 \end{array}}$$

From b.c. (i), (ii)

$\Rightarrow$ $C = 0, -n^2, \; n = 1, 2, 3, \ldots$

$G_0(x) = 1, \; G_n(x) = \cos(nx)$

observing b.c. (iii), (iv), only $n=3$ matters.

For $n=3$, $\quad y^2\ddot{H}_3 - 2y\dot{H}_3 + 5H_3 = 9H_3$

$$\Rightarrow y^2\ddot{H}_3 - 2y\dot{H}_3 - 4H_3 = 0$$

Let $H_3 \sim y^p \Rightarrow y^2 \cdot p(p-1)y^{p-2} - 2y \cdot p y^{p-1} - 4y^p = 0$

$$\Rightarrow p^2 - 3p - 4 = 0 \Rightarrow \underline{p = 4, -1}$$

$$\Rightarrow H_3(y) = A_3 y^4 + B_3 y^{-1}$$

Full solution: $u(x,y) = (A_3 y^4 + B_3 y^{-1})\cos(3x)$

From b.c. (iii), $\quad A_3 + B_3 = 3$ —— ①

From b.c. (iv), $\quad 16A_3 + \frac{1}{2}B_3 = 17$ —— ②

solving ①, ② $\Rightarrow A_3 = 1, \; B_3 = 2$

Full solution is $u(x,y) = \left(y^4 + \frac{2}{y}\right)\cos(3x)$

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