Q3

By Fourier series expansion:

$$u(x,t) = \sum_{n=-\infty}^{\infty} C_n(t) e^{inx}$$

$\Rightarrow$

$$\dot{C}_n = \left(in^5 t - i4n^3 t - n^2 + 1\right) C_n$$

observing the b.c., only $n=0, 1, 2$ matter (also $n=-1, -2$)

Never need to process

$n=0$:

$$\dot{C}_0 = C_0$$

$$C_0(t) = C_0(0) e^t = e^t$$

$n=1$:

$$\dot{C}_1(t) = -i3t\, C_1$$

$$\Rightarrow \quad C_1(t) = C_1(0) e^{-i3t^2/2} = \frac{1}{2i} e^{-i3t^2/2}$$

$$u(x,0) = \sum_{n=-\infty}^{\infty} C_n(0) e^{inx}$$

$$\parallel$$

$$1 + \sin(x) + \cos(2x)$$

$$\parallel$$

$$1 + \left\{\left[\frac{1}{2i} e^{ix} + \frac{1}{2} e^{i2x}\right] + c.c.\right\}$$

$$\Rightarrow \quad C_0(0) = 1$$

$$C_1(0) = \frac{1}{2i} \qquad \left(C_{-1}(0) = \frac{-1}{2i}\right.$$

$$C_2(0) = \frac{1}{2} \qquad \left.C_{-2}(0) = \frac{1}{2}\right)$$

Never need to process

All other $C_n(0) = 0$

$n=2$: $\dot{C}_2 = -3C_2 \Rightarrow C_2(t) = C_2(0) e^{-3t} = \frac{1}{2} e^{-3t}$

Full solution:

$$u(x,t) = C_0(t) + \left\{\left[C_1(t) e^{ix} + C_2(t) e^{i2x}\right] + c.c.\right\}$$

$$= e^t + \left\{\left[\frac{1}{2i} e^{i(x - 3t^2/2)} + \frac{1}{2} e^{-3t} e^{i2x}\right] + c.c.\right\}$$

$$= e^t + \sin\left(x - \frac{3t^2}{2}\right) + e^{-3t}\cos(2x) \qquad \#$$

* See <u>Lecture 17</u> for detailed derivation for this term.

Q4 By Fourier series expansion:

$$u(x,t) = \sum_{n=-\infty}^{\infty} C_n(t)\, e^{inx}$$

$$\Rightarrow \ddot{C}_n = (n^4 - 2n^2 + 1)\, C_n$$

From the b.c.'s, only $n=0, 1$ are relevant (also $n=-1$)

No need to process

$n=0$:
$$\ddot{C}_0 = C_0$$

$$\Rightarrow C_0(t) = A_0 \cosh(t) + B_0 \sinh(t)$$

$$\Rightarrow \dot{C}_0(t) = A_0 \sinh(t) + B_0 \cosh(t)$$

$C_0(0) = 0 \Rightarrow \underline{A_0 = 0}$ $\quad \dot{C}_0(0) = 1 \Rightarrow \underline{B_0 = 1}$

$$\Rightarrow \underline{\underline{C_0(t) = \sinh(t)}}$$

$$u(x,0) = \sum_{n=-\infty}^{\infty} C_n(0)\, e^{inx}$$

$//$

$\cos(x)$

No need to process

$$\Rightarrow \begin{cases} C_1(0) = \frac{1}{2} & \left(C_{-1}(0) = \frac{1}{2}\right) \\ C_0(0) = 0 \end{cases}$$

$$u_t(x,0) = \sum_{n=-\infty}^{\infty} \dot{C}_n(0)\, e^{inx}$$

$//$

$1 + \sin(x)$

No need to process

$$\Rightarrow \begin{cases} \dot{C}_0(0) = 1 \\ \dot{C}_1(0) = \frac{1}{2i} & \left(\dot{C}_{-1}(0) = \frac{1}{2i}\right) \end{cases}$$

All other $C_n(0)$, $\dot{C}_n(0)$ are zero.

$n=1$: $\ddot{C}_1 = 0 \Rightarrow C_1(t) = A_1 t + B_1$

$\dot{C}_1(t) = A_1$

$\dot{C}_1(0) = \frac{1}{2i} \Rightarrow A_1 = \frac{1}{2i}$, $\quad C_1(0) = \frac{1}{2} \Rightarrow B_1 = \frac{1}{2}$, so, $\underline{\underline{C_1(t) = \frac{1}{2i} t + \frac{1}{2}}}$

Full solution:
$$u(x,t) = C_0(t) + \left\{ C_1(t)\, e^{ix} + c.c. \right\}$$
$$= \sinh(t) + \left\{ \left(\frac{1}{2i} t + \frac{1}{2}\right) e^{ix} + c.c. \right\}$$
$$= \sinh(t) + \underbrace{\cos(x) + t\sin(x)}_{\text{detail}} \qquad \#$$

$$\left\{ \left(\frac{1}{2i} t + \frac{1}{2}\right) e^{ix} + c.c. \right\} = 2 \cdot \mathrm{Re} \left\{ \left(\frac{1}{2i} t + \frac{1}{2}\right) e^{ix} \right\}$$
$$= 2 \cdot \mathrm{Re} \left\{ \left(-\frac{t}{2} i + \frac{1}{2}\right) (\cos(x) + i \sin(x)) \right\}$$
$$= 2 \cdot \mathrm{Re} \left\{ \frac{1}{2}\cos(x) + \frac{t}{2}\sin(x) + i\,[\text{imag part}] \right\}$$
$$= \cos(x) + t\sin(x)$$

Q5

By Fourier series expansion:

$$u(x,t) = \sum_{n=-\infty}^{\infty} C_n(t)\, e^{inx}$$

$$\Rightarrow \quad \ddot{C}_n - 2\dot{C}_n + (-n^2+1)\,C_n = 0$$

From the b.c.'s, only $n=1, 2$ matter

$n=1$: $\ddot{C}_1 - 2\dot{C}_1 = 0$

Let $C_1(t) \sim e^{\alpha t} \Rightarrow \alpha^2 - 2\alpha = 0$

$\Rightarrow \alpha = 0, 2$, so

$$C_1(t) = A_1 + A_2 e^{2t}$$

$$\dot{C}_1(t) = 2A_2 e^{2t}$$

$\dot{C}_1(0) = 0 \Rightarrow A_2 = 0$, $C_1(0) = \frac{1}{2} \Rightarrow A_1 = \frac{1}{2}$

$$\Rightarrow C_1(t) = \frac{1}{2}$$

$$u(x,0) = \sum_{n=-\infty}^{\infty} C_n(0)\, e^{inx} = \cos(x)$$

$$\Rightarrow \begin{cases} C_1(0) = \frac{1}{2} \\ C_2(0) = 0 \end{cases}$$

$$u_t(x,0) = \sum_{n=-\infty}^{\infty} \dot{C}_n(0)\, e^{inx} = \sin(2x)$$

$$\Rightarrow \begin{cases} \dot{C}_1(0) = 0 \\ \dot{C}_2(0) = \frac{1}{2i} \end{cases}$$

All other $C_n(0)$, $\dot{C}_n(0)$ are zero

$n=2$: $\ddot{C}_2 - 2\dot{C}_2 - 3C_2 = 0$ Let $C_2(t) \sim e^{\alpha t} \Rightarrow \alpha^2 - 2\alpha - 3 = 0$

$\Rightarrow \alpha = -1, 3$

$$\Rightarrow C_2(t) = A_2 e^{-t} + B_2 e^{3t}$$

$$\dot{C}_2(t) = -A_2 e^{-t} + 3B_2 e^{3t}$$

$$\left.\begin{aligned} C_2(0) = 0 &\Rightarrow A_2 + B_2 = 0 \\ \dot{C}_2(0) = \tfrac{1}{2i} &\Rightarrow -A_2 + 3B_2 = \tfrac{1}{2i} \end{aligned}\right\} \Rightarrow A_2 = \frac{-1}{8i},\ B_2 = \frac{1}{8i}$$

$$C_2(t) = \frac{1}{8i}\left(e^{3t} - e^{-t}\right)$$

Full Solution:

$$u(x,t) = \left\{C_1(t)\, e^{ix} + C_2(t)\, e^{i2x}\right\} + \text{c.c.}$$

$$= \left\{\frac{1}{2} e^{ix} + \frac{1}{8i}\left(e^{3t} - e^{-t}\right) e^{i2x}\right\} + \text{c.c.}$$

$$= \cos(x) + \frac{1}{4}\left(e^{3t} - e^{-t}\right)\sin(2x)$$

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