Q1 By separation of variables of the homogeneous sub-system, we obtain the b.b. in x-direction as $\{1, \cos(nx), n=1,2,3,\dots\}$

$$\Rightarrow \begin{cases} u(x,t) = a_0(t) + \sum_{n=1}^{\infty} a_n(t)\cos(nx) \\ Q(x,t) = q_0(t) + \sum_{n=1}^{\infty} q_n(t)\cos(nx) \end{cases}$$

visual inspection: $a_0(0)=1$, $a_3(0)=1$

all other $a_n(0)=0$

visual inspection: $q_0(t)=e^t$, $q_1(t)=\sin(t)$

all other $q_n(t)=0$

plugging back to the full PDE, we have

$$\frac{da_n}{dt} = -n^2 a_n + a_n + q_n$$

* only $n=0, 1, 3$ matter

$n=0$: $\frac{da_0}{dt} = a_0 + e^t \Rightarrow a_0(t) = \underbrace{a_0(0)}_{=1} e^t + \int_0^t e^{\hat{t}} e^{(t-\hat{t})} d\hat{t} = (1+t)e^t$

$n=1$: $\frac{da_1}{dt} = \sin(t) \Rightarrow a_1(t) = \underbrace{a_1(0)}_{0} + \int_0^t \sin(t)dt = 1-\cos(t)$

$n=3$: $\frac{da_3}{dt} = -8a_3 \Rightarrow a_3(t) = \underbrace{a_3(0)}_{1} e^{-8t} = e^{-8t}$

Full solution:

$$u(x,t) = a_0(t) + a_1(t)\cos(x) + a_3(t)\cos(3x)$$

$$= (1+t)e^t + [1-\cos(t)]\cos(x) + e^{-8t}\cos(3x)$$

#

Q2 By separation of variables of the homogeneous sub-system, we obtain the b.b. in x-direction as $\{\sin(nx),\ n=1,2,3,\dots\}$

$$\Rightarrow\ u(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin(nx) \Rightarrow$$

visual inspection:
From b.c. (iii),
$a_1(0) = 1$
From b.c. (iv),
$\dot{a}_1(0) = 1$.
All other $a_n(0)$, $\dot{a}_n(0)$ are zero.

$\Downarrow$

* only $\underline{\underline{n=1}}$ matters

plugging it back to the full PDE, we have:

$$\frac{d^2 a_n}{dt^2} = -n^2 a_n + q_n$$

$$Q(x,t) = \sum_{n=1}^{\infty} q_n(t)\sin(nx)$$

visual inspection:

$q_1(t) = -1$.

All other $q_n(t) = 0$

For $n=1$: $\ddot{a}_1 = -a_1 - 1$ Let $b \equiv a_1 + 1$

$$\Rightarrow \ddot{b} = -b \Rightarrow b(t) = A\cos(t) + B\sin(t)$$

$$\Rightarrow a_1(t) = A\cos(t) + B\sin(t) - 1$$

$a_1(0) = 1 \Rightarrow A - 1 = 1 \Rightarrow A = 2$

$\dot{a}_1(0) = 1 \Rightarrow B = 1$

$\dot{a}_1(t) = -A\sin(t) + B\cos(t)$

Full solution: $u(x,t) = a_1(t)\sin(x)$

$$= [2\cos(t) + \sin(t) - 1]\sin(x)$$

#

Q3

First, attempt to find the steady solution, $u_s(x)$.

$$u_s'' + u_s + 1 = 0 \qquad u_s(0) = -1 \;,\; u_s\left(\tfrac{\pi}{2}\right) = 0$$

$$\Rightarrow u_s'' = -(u_s + 1) \qquad \text{Let } V \equiv u_s + 1$$

$$\Rightarrow V'' = -V \Rightarrow V(x) = A\cos(x) + B\sin(x)$$

$$\Rightarrow u_s(x) = A\cos(x) + B\sin(x) - 1$$

$$u_s(0) = -1 \Rightarrow A - 1 = -1 \Rightarrow A = 0$$

$$u_s\left(\tfrac{\pi}{2}\right) = 0 \Rightarrow B - 1 = 0 \Rightarrow B = 1$$

$$\Rightarrow u_s(x) = \underline{\sin(x) - 1}$$

Let $\hat{u}(x,t) \equiv u(x,t) - u_s(x)$, then the system becoms

$$\boxed{\frac{\partial \hat{u}}{\partial t} = \frac{\partial^2 \hat{u}}{\partial x^2} + \hat{u} \;,\quad \hat{u}(0,t) = 0,\; \hat{u}\left(\tfrac{\pi}{2}, t\right) = 0,\; \hat{u}(x,0) = \sin(2x)}$$

Solving this system with standard technique, we have

$$\hat{u}(x,t) = \sum_{n=1}^{\infty} a_n \sin(2nx)\, e^{[-(2n)^2+1]t}$$

From the last b.c., only $n=1$ matter, and $a_1 = 1$.

$\Rightarrow$ Full solution for $\hat{u}$ is $\hat{u}(x,t) = \sin(2x)\, e^{-3t}$

$\Rightarrow$ Full solution for u is

$$u(x,t) = \hat{u}(x,t) + u_s(x)$$

$$= \sin(2x)\, e^{-3t} + \sin(x) - 1$$

#

Q4 By Fourier series expansion:

$$\begin{cases} u(x,t) = \sum_{n=-\infty}^{\infty} C_n(t)\, e^{inx} \\ Q(x,t) = \sum_{n=-\infty}^{\infty} q_n(t)\, e^{inx} \end{cases}$$

visual inspection:

$q_0(t) = \sin(t)$, All other $q_n(t) = 0$

$C_1(0) = \frac{1}{2}$ (also, $C_{-1}(0) = \frac{1}{2}$)

$\dot{C}_0(0) = 1$

All other $C_n(0)$, $\dot{C}_n(0)$ are zero

only $n = 0, 1$ matter (and -1)

plugging back to the full PDE, we have

$$\ddot{C}_n = inC_n - n^2 C_n - in^3 \dot{C}_n + q_n$$

$n=0$: $\ddot{C}_0 = q_0 = \sin(t)$

$$\Rightarrow \dot{C}_0(t) = \dot{C}_0(0) + \int_0^t \sin(t)\,dt$$

$$= \dot{C}_0(0) + [1 - \cos(t)]$$

$$= 2 - \cos(t)$$

$$\Rightarrow C_0(t) = \underbrace{C_0(0)}_{0} + \int_0^t [2 - \cos(t)]\,dt = 2t - \sin(t)$$

$n=1$: $\ddot{C}_1 = -C_1 \Rightarrow C_1(t) = A\cos(t) + B\sin(t)$

$$\dot{C}_1(t) = -A\sin(t) + B\cos(t)$$

$C_1(0) = \frac{1}{2} \Rightarrow A = \frac{1}{2}$ $\quad \dot{C}_1(0) = 0 \Rightarrow B = 0$

$$\Rightarrow C_1(t) = \frac{1}{2}\cos(t)$$

Full solution: $u(x,t) = C_0(t) + \{ C_1(t) e^{ix} + \text{c.c.} \}$

$$= [2t - \sin(t)] + \{ \tfrac{1}{2}\cos(t) e^{ix} + \text{c.c.} \}$$

$$= 2t - \sin(t) + \cos(t)\cos(x)$$

#