Q1 $\rightarrow$ $\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = e^{-t}(xu+1)$

By MOC :

$$\frac{dx}{dt} = x \Rightarrow x(t) = x(0)e^{t} \quad \text{—— ①}$$

$$\frac{du}{dt} = e^{-t}(xu+1) \overset{\Downarrow}{=} x(0)u + e^{-t} \quad \text{—— ②}$$

The solution for ② is

$$u(t) = u(0)e^{x(0)t} + \int_0^t e^{-\hat{t}} e^{x(0)(t-\hat{t})}\, d\hat{t}$$

$\boxed{\text{From b.c., } u(0)=0}$

$$= e^{x(0)t}\int_0^t e^{-(x(0)+1)\hat{t}}\, d\hat{t}$$

$$= \frac{1-e^{-(x(0)+1)t}}{x(0)+1}\cdot e^{x(0)t}$$

$\boxed{x(0) = x(t)e^{-t}}$

$$= \frac{e^{x(t)\cdot te^{-t}} - e^{-t}}{x(t)e^{-t}+1}$$

$$\Rightarrow \text{Full solution: } u(x,t) = \frac{e^{xte^{-t}} - e^{-t}}{xe^{-t}+1} \quad \text{—— } (\star) \quad \#$$

Note: when $x(0) = -1$, the integral $\int_0^t e^{-(x(0)+1)\hat{t}}\, d\hat{t} = t$,

and the solution is $u(t) = te^{x(0)t} = te^{-t}$.

At the same time, $x(0) = -1$ means $x(t) = -e^{t}$.

So, the full solution is $u(x,t) = te^{-t}$

when $x = -e^{t}$. This resolves the

apparent singularity in $(\star)$ when $x = -e^{t}$.

(The same conclusion can be reached by

applying L'Hospital's rule to $\star$ at $x = -e^{t}$)

Q2

By MOC: $\dfrac{dx}{dt} = u = u(0) + 2t \Rightarrow x(t) = x(0) + u(0)t + t^2$ — ①

$\dfrac{du}{dt} = 2 \Rightarrow u(t) = u(0) + 2t$ — ②

If $x(0) > -1 \Rightarrow u(0) = 1 \Longrightarrow u(t) = 1 + 2t$

$\Downarrow$

$x(t) = x(0) + t + t^2$

$x(t) > -1 + t + t^2$

$$\boxed{\begin{aligned} &u(x,t) = 1 + 2t \\ &\text{if } \quad x > -1 + t + t^2 \end{aligned}}$$

If $x(0) \le -1 \Rightarrow u(0) = -\dfrac{1}{x(0)} \Rightarrow u(t) = -\dfrac{1}{x(0)} + 2t$

$\Downarrow$

$x(t) = x(0) - \dfrac{t}{x(0)} + t^2$

$x(t) \le -1 + t + t^2$

$\Downarrow$

$[x(0)]^2 + [t^2 - x(t)] - t = 0$

$\Downarrow$

$$x(0) = \frac{(x(t) - t^2) \pm \sqrt{[t^2 - x(t)]^2 + 4t}}{2}$$

* The "+" root is not valid as it violates $x(0) \le -1$

Together, full solution

$$\boxed{\begin{aligned} &u(x,t) = -\frac{2}{(x - t^2) - \sqrt{(t^2 - x)^2 + 4t}} + 2t \\ &\text{if } \quad x \le -1 + t + t^2 \end{aligned}}$$

Q3

By MOC, $\dfrac{dx}{dt} = u \quad —①$

$\dfrac{du}{dt} = x + 1 \quad —②$

combining ①+②: $\dfrac{d(x+u)}{dt} = (x+u)+1$

Let $v \equiv x+u \Rightarrow \dfrac{dv}{dt} = v+1 \Rightarrow v(t) = v(0)e^t + e^t - 1$

$\Downarrow$

$u(t) + x(t) = \underbrace{[u(0)+x(0)]}_{0}e^t + e^t - 1$

From b.c., $u(0) = -x(0) \Longrightarrow 0$

$\Rightarrow \quad u(t) = -x(t) + e^t - 1$

Full solution is $u(x,t) = -x + e^t - 1$ #

Q4 : $\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) u = x$

decompose the equation into 2 first-order PDEs:

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) \underbrace{\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u}_{= W} = x$$

$\Rightarrow$ $W(x,t) \equiv u_t(x,t) - u_x(x,t)$

$\Downarrow$

$W(x,0) = u_t(x,0) - u_x(x,0) = 0$

This is the b.c. for ①

$$\Rightarrow \begin{cases} \frac{\partial w}{\partial t} + \frac{\partial w}{\partial x} = x & \text{— ①} \\ \frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} = w & \text{— ②} \end{cases}$$

Solve ① by MOC: $\frac{dx}{dt} = 1 \Rightarrow x(t) = x(0) + t$

$$\frac{dw}{dt} = x = x(0) + t \Rightarrow w(t) = \overset{0}{w(0)} + x(0)t + \frac{t^2}{2}$$

$$= (x(t) - t)t + \frac{t^2}{2}$$

$$w(x,t) = tx - \frac{t^2}{2}$$

Solve ② by MOC:

$$\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} = tx - \frac{t^2}{2}$$

$$\frac{dx}{dt} = -1 \Rightarrow x(t) = x(0) - t$$

$$\frac{du}{dt} = tx - \frac{t^2}{2} = tx(0) - t^2 - \frac{t^2}{2}$$

$$\Rightarrow u(t) = u(0) + \frac{t^2}{2}x(0) - \frac{t^3}{2}$$

$$= 1 + \frac{t^2}{2}(x(t) + t) - \frac{t^3}{2}$$

$$= 1 + \frac{t^2}{2}x(t)$$

From b.c. $u(0) = 1$

$\Rightarrow$ Full solution is

$$u(x,t) = 1 + \frac{t^2 x}{2}$$

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