

PLEASE READ THIS PAGE FIRST

A statement of collaboration is required. If no collaboration occurred, write “No collaboration”. A violation of the rule(s) given in this page will be considered a violation of ASU's Academic Integrity Policy.

Rules on collaboration for homework:

(1) Collaboration is not allowed unless all involved follow rules (2)-(3) and unless the extent of collaboration is properly disclosed in a statement; See additional instruction below for the required content of the statement.

(2) For each assignment, each person can have maximum of one collaborator. Be aware that a collaborator's collaborator counts as a collaborator. For example, if Alice collaborates with Bob and Bob collaborates with Charles, Charles counts as a collaborator of Alice. All three violate the rule. In other words, collaboration can only be carried out within an isolated "team of two". Please talk to a potential collaborator to ensure that this rule is not violated before establishing any collaboration.

(3) In a legitimate collaboration, each individual must make a non-negligible contribution to the collaborative effort. Taking the solution or code from another student without making a contribution to it is not allowed. To certify that a collaboration is legitimate, the submitter's contribution to the collaborative effort must be documented in the statement of collaboration.

Statement of collaboration

This statement is mandatory and must be placed in the beginning of the first page of your submission. If no collaboration occurred, simply state "**No collaboration**". This implies that the person submitting the work *has not helped anyone or received help from anyone* in the process of completing the assignment. If collaboration occurred, provide the name of collaborator (only one allowed), a list of the task(s) on which collaboration occurred, and a description of the contribution by the submitter to the collaborative effort. Example:

Name of collaborator: <i>Joe Smith</i>	
Task	Contribution to collaborative effort
<i>Problem 1</i>	<i>Developed codes with collaborator</i>
<i>Problem 3</i>	<i>Compared mathematical derivation with collaborator</i>

MAE/MSE 502, Spring 2023 Homework #1
(1 point \approx 1% of the total score for the course.)

A statement of collaboration is required. Please include computer codes (if any are used) in your submission.

Problem 1 (4.5 points)

(a) For $u(x, t)$ defined on the domain of $0 \leq x \leq 1$ and $t \geq 0$, solve the PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

with the boundary conditions (u_x denotes $\partial u / \partial x$),

(i) $u_x(0, t) = 0$, (ii) $u_x(1, t) = 0$, (iii) $u(x, 0) = P(x)$,

where

$$P(x) = 4x^5 - 5x^4 + 1 + [\sin(\pi x)]^8 .$$

Plot the solution, $u(x, t)$, as a function of x at $t = 0, 0.005, 0.025$, and 0.25 . Please collect all four curves in a single plot.

(b) Repeat (a) but now solve the system with the second boundary condition changed to

(ii) $u(1, t) = 0$.

The 1st and 3rd boundary conditions remain the same as in Part (a). Plot the solution, $u(x, t)$, as a function of x at $t = 0, 0.005, 0.025$, and 0.25 . Please collect all four curves in a single plot.

Note for Problem 1:

(i) We expect the solution to be expressed as an infinite series. A truncation of the infinite series is needed to numerically compute the series in order to plot the solution. It is your job to determine the appropriate number of terms to keep. As a useful measure, the solution at $t = 0$ should match the given initial state in the 3rd boundary condition. If they do not match, either the solution is wrong or more terms need to be retained in the series. This remark applies to all future homework problems that require the evaluation of an infinite series.

(ii) For the evaluation of the expansion coefficients in the infinite series, there is no need to carry out the integrations analytically. (If you wish to do so, please feel free to use an online integrator or similar software such as Wolfram Alpha. No need to do it by hand.) It is perfectly acceptable (actually, **recommended** for this problem) to evaluate the integrals numerically using Matlab. See Additional Note in the last page for an example of using Matlab to numerically evaluate an integral.

Problem 2 (2 points)

For $u(x,t)$ defined on the domain of $0 \leq x \leq \pi$ and $t \geq 0$, solve the PDE,

$$\frac{\partial u}{\partial t} = \left(\frac{2}{1+t}\right) \frac{\partial^2 u}{\partial x^2} + \left(\frac{8}{1+t}\right) u ,$$

with the boundary conditions,

$$(i) u_x(0, t) = 0 , (ii) u_x(\pi, t) = 0, (iii) u(x, 0) = 1 + \cos(2x)$$

For this problem, we expect a closed-form exact solution with only a finite number of terms and without any unevaluated integral. Expect a deduction if the requirement is not satisfied.

Problem 3 (2 points)

For $u(x,t)$ defined on the domain of $0 \leq x \leq 2023$ and $t \geq 0$, solve the PDE,

$$e^t \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \pi^2 u ,$$

with the boundary conditions,

$$(i) u(0, t) = 0 , (ii) u(2023, t) = 0, (iii) u(x, 0) = \sin(\pi x) + \sin(2\pi x) .$$

For this problem, we expect a closed-form exact solution with only a finite number of terms and without any unevaluated integrals. Expect a deduction if the requirement is not satisfied.

Problem 4 (1 point)

For $u(x,t)$ defined on the domain of $0 \leq x \leq 5$ and $t \geq 0$, consider the system of PDE,

$$\frac{\partial u}{\partial t} = 100 \frac{\partial^2 u}{\partial x^2} ,$$

with the boundary conditions (u_x denotes $\partial u / \partial x$),

$$(i) u_x(0, t) = 3 , (ii) u(5, t) = 16 , (iii) u(x, 0) = x^2 + 3x - 24 .$$

Find the **steady solution** for the system. NOTE: Steady solution is the only deliverable for this problem. No need to find the full solution.

Problem 5 (0.5 point)

The 1-D Heat equation in its dimensional form is

$$\frac{\partial u}{\partial \hat{t}} = K \frac{\partial^2 u}{\partial \hat{x}^2} , 0 \leq \hat{x} \leq L \quad (L \text{ is the length of the "metal rod", in meters) and } \hat{t} \geq 0 , \quad (1)$$

where \hat{t} and \hat{x} are time in seconds and distance in meters, K is thermal diffusivity in m^2/s , and u is temperature. In our class, we usually consider the non-dimensionalized version of (1),

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 \leq x \leq 1 \quad \text{and} \quad t \geq 0, \quad (2)$$

where x is related to \hat{x} by $\hat{x} = Lx$. In (2), time is also non-dimensionalized by $\hat{t} = Tt$, where T is a certain dimensional time scale, and so on.

(a) In order to claim that the non-dimensionalized system, (1), is equivalent to its dimensional counterpart, (2), the three parameters K , L , and T must satisfy a unique relation. First, find out what this relation is. (To answer part (b) and (c), it is useful to write the relation as $T = f(K, L)$.)

(b) Suppose that the non-dimensionalized Heat equation, (2), is used to model the real world problem of heat transfer along a metal rod that is 1 meter long and made of copper ($K \approx 0.0001 \text{ m}^2/\text{s}$), what would be the actual time, in seconds, that $t = 0.01$ corresponds to in that problem?

(c) Same as (b), but suppose that (2) describes heat transfer along a wooden stick that is 0.3 meter long and made of pine wood ($K \approx 10^{-7} \text{ m}^2/\text{s}$), what would be the actual time, in seconds, that $t = 0.01$ corresponds to?

Additional Note: Using Matlab to numerically evaluate an integral

One of the simplest Matlab functions for this purpose is **trapz**. It uses the trapezoidal method to evaluate an integral. For example, to numerically evaluate

$$I = \int_0^1 \sin(x) dx$$

with $\Delta x = 0.01$, we first construct the discretized arrays of the coordinate points and the values of the integrand at those points. We then call **trapz** with those two arrays as the input to complete the integration. The Matlab code is very simple:

```
x = [0:0.01:1];
y = sin(x);
Integ = trapz(x,y)
```

The outcome can be validated with the analytic result of $I = 1 - \cos(1)$. When using **trapz**, it is important to use a sufficient grid resolution to avoid "aliasing errors". It requires a minimum of 5 points (10 points are recommended) to resolve one cycle of a sinusoidal wave. In the example above, 100 grid points are used to resolve less than 1/6 of a cycle, which is sufficient. If the integrand in the example is changed to **y = sin(500*x)**, the result of numerical integration will be entirely wrong. The remedy would be to also increase the grid resolution. For example, change the grid to **x = [0:0.0001:1]**.