Q2

By separation of variables, $u \sim G(x)\,H(t)$

$$\frac{1+t}{2}\frac{\dot{H}}{H} - 4 = \boxed{\frac{G''}{G} = c \qquad \begin{matrix} G'(0)=0 \\ G'(\pi)=0 \end{matrix}} \Rightarrow c = 0,\ -n^2 \quad n=1,2,3,\dots$$

$G_0(x) = 1$

$G_n(x) = \cos(nx), \quad n=1,2,3,\dots$

observing b.c.(III),
only $c=0,\ -4$ matter.

$$c=0 \Rightarrow \frac{1+t}{2}\frac{\dot{H}_0}{H_0} - 4 = 0$$

$$\Rightarrow \frac{\dot{H}_0}{H_0} = \frac{8}{1+t} \Rightarrow H_0(t) = H_0(0)\, e^{8\int_0^t \frac{dt}{1+t}} = \underbrace{H_0(0)}_{\text{OK to set to 1}}\,(1+t)^8$$

$$\begin{matrix} c=-4 \\ (n=2) \end{matrix} \Rightarrow \frac{1+t}{2}\frac{\dot{H}_2}{H_2} - 4 = -4 \Rightarrow \dot{H}_2 = 0 \qquad H_2(t) = \underbrace{H_2(0)}_{\text{OK to set to 1.}}$$

Full solution: $u(x,t) = a_0\, G_0(x) H_0(t) + a_2\, G_2(x) H_2(t)$

$$= a_0 (1+t)^8 + a_2 \cos(2x)$$

From b.c. (III) $\Rightarrow$ $a_0 = 1,\ a_2 = 1.$

Full solution: $u(x,t) = (1+t)^8 + \cos(2x)$

#

Q3

By separation of variables, $u \sim G(x)H(t)$,

$$e^t \frac{\dot{H}}{H} - \pi^2 = \boxed{\frac{G''}{G} = c \qquad \begin{array}{l} G(0)=0 \\ G(2023)=0 \end{array}} \Rightarrow \cancel{c_n = -(n\pi)^2} \quad c_n = -\left(\frac{n\pi}{2023}\right)^2 \quad n=1,2,3,\dots$$

$$G_n(x) = \sin\left(\frac{n\pi x}{2023}\right) \quad n=1,2,3,\dots$$

observing b.c.(III), only $n = 2023, 4046$ matter

$$n=2023: \quad e^t \frac{\dot{H}_{2023}}{H_{2023}} - \pi^2 = -\pi^2 \Rightarrow H_{2023}(t) = \underbrace{H_{2023}(0)}_{\text{OK to set to 1}}$$

$$n=4046: \quad e^t \frac{\dot{H}_{4046}}{H_{4046}} - \pi^2 = -4\pi^2$$

$$\Rightarrow \frac{\dot{H}_{4046}}{H_{4046}} = -3\pi^2 e^{-t} \Rightarrow H_{4046}(t) = H_{4046}(0)\, e^{-3\pi^2 \int_0^t e^{-t} dt}$$

$$= \underbrace{H_{4046}(0)}_{\text{OK to set to 1}} e^{-3\pi^2(1-e^{-t})}$$

Full solution: $u(x,t) = a_{2023}\, G_{2023}(x) H_{2023}(t) + a_{4046}\, G_{4046}(x) H_{4046}(t)$

$$= a_{2023} \sin(\pi x) + a_{4046} \sin(2\pi x)\, e^{-3\pi^2(1-e^{-t})}$$

From b.c.(III), $a_{2023} = 1$, $a_{4046} = 1$

$\Rightarrow$ Full solution is

$$u(x,t) = \sin(\pi x) + \sin(2\pi x)\, e^{-3\pi^2(1-e^{-t})}$$

#