

MAE/MSE 502, Spring 2023 Homework #2

A statement of collaboration is required. Please include computer codes (if any are used) in your submission.

Problem 1 (4 points)

For $u(x,y)$ defined on the square domain of $0 \leq x \leq 1$ and $0 \leq y \leq 1$, solve the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

with the boundary conditions,

$$(i) u(0, y) = 0 \quad (ii) u(1, y) = 0 \quad (iii) u(x, 0) = \sin(\pi\sqrt{x}) \quad (iv) u(x, 1) = 2(x - x^5) .$$

Make a contour plot of the solution, $u(x, y)$. See Additional Note in the last page for an example of using Matlab to make a contour plot. The recommended contour interval for the plot is 0.05. (This is different from what is given in the example in Additional Note. Please make adjustments accordingly.) For this problem, we expect the solution to be expressed as an infinite series. Please see the remark below HW1-Q1 on how to truncate the series and numerically compute the expansion coefficients.

Problem 2 (3 points)

For $u(x,y)$ defined on the square domain of $0 \leq x \leq 1$ and $0 \leq y \leq 1$, consider the PDE

$$9 \frac{\partial^2 u}{\partial x^2} + 16 \frac{\partial^2 u}{\partial y^2} = 0$$

with the boundary conditions (u_x and u_y denote $\partial u/\partial x$ and $\partial u/\partial y$, respectively)

$$(i) u_x(0, y) = 1 \quad (ii) u_x(1, y) = 1 + \cos(3\pi y) \quad (iii) u_y(x, 0) = 0 \quad (iv) u_y(x, 1) = 0$$

(a) Test the solvability condition on the system. Based on it, which of the following is true?

(I) The system has no solution. (II) The system has a unique solution. (III) The system has multiple solutions. If your answer is (I), no need to proceed further. Otherwise, proceed to Part (b). **Note: You must demonstrate the procedure of testing solvability condition to receive credit for Part (a), regardless of whether your solution for Part (b) is correct.**

(b) Find the solution(s) of the system. If your answer for Part (a) is (III), please clearly write out what those multiple solutions are. For this part of the problem, we expect a closed-form solution that consists of only a finite number of terms and without any unevaluated integrals. Expect a deduction if the requirement is not met.

Problem 3 (2.5 points)

For $u(x, y)$ defined on the domain of $0 \leq x \leq \pi$ and $0 \leq y \leq \ln(2)$, solve the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - 4 \frac{\partial u}{\partial y} + 7u = 0,$$

with the boundary conditions

$$(i) u(0, y) = 0 \quad (ii) u(\pi, y) = 0 \quad (iii) u_y(x, 0) = 2 \sin(2x) \quad (iv) u(x, \ln 2) = 2 \sin(2x) .$$

(The “ \ln ” in the above statement is natural log. Note that the 3rd boundary condition is imposed on the derivative of u .) For this problem, we expect a closed-form analytic solution that consists of only a finite number of terms and without any unevaluated integral. Expect a deduction if the requirement is not met.

Problem 4 (1.5 points)

For $u(x,y)$ defined on the square domain of $0 \leq x \leq 1$ and $0 \leq y \leq 1$, solve the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

with the boundary conditions

$$(i) u(0, y) = 3 \quad (ii) u(1, y) = 3 + \sin(2023 \pi y) \quad (iii) u(x, 0) = 3 \quad (iv) u(x, 1) = 3.$$

For this problem, we expect a closed-form analytic solution that consists of only a finite number of terms and without any unevaluated integral. Expect a deduction if the requirement is not met.

[Note: This problem is inspired by questions raised by students in Lecture 11: “What if the values of u given in the boundary conditions are not zero at the corners of the square domain?” Here, we have the simplest system with that property. In this case, directly “splitting” the original system into multiple systems (with some homogeneous boundary conditions) will not work, because each of the new systems will have discontinuities in the boundary conditions. Nevertheless, the problem can actually be solved with a simple “shift” in u . It is fine to keep the procedure brief.]

Additional Note: Using Matlab to make a contour plot

The following Matlab code makes a contour plot for $u(x,y) = \sin(2\pi x) + x \cos(3\pi y)$ over the domain of $0 \leq x \leq 1$ and $0 \leq y \leq 1$, using a contour interval of 0.2 over the range of $-2 \leq u \leq 2$. Selected contours with $u = -1.6 + 0.4p$, $p = 1, 2, 3, \dots$, are labeled. The contour plot is shown in the next page.

It is essential to provide the coordinates of the grid ($x2d$ and $y2d$ in this example) as the input for the contour function. Without this piece of information, Matlab would not know the grid spacing and the correct directions of x and y . Be aware that the setting used in this example might not be what is recommended for the actual contour plot in Problem 1. For that problem, please adjust the contour interval as needed.

```
x = [0:0.01:1]; y = [0:0.01:1];
for m = 1:length(x)
    for n = 1:length(y)
        u(m,n) = sin(2*pi*x(m)) + x(m)*cos(3*pi*y(n));
        x2d(m,n) = x(m); y2d(m,n) = y(n);
    end
end
[C h] = contour(x2d,y2d,u, [-2:0.2:2], 'LineWidth', 2);
clabel(C,h, [-1.6:0.4:1.6]);
xlabel('x'); ylabel('y')
```

