Q2 (b)

By separation of variables, $u \sim G(x)\,H(y)$,

$$\Rightarrow \quad -\frac{9}{16}\frac{G''}{G} = \boxed{\frac{\ddot{H}}{H} = c \qquad \dot{H}(0)=0,\ \dot{H}(1)=0} \Rightarrow c = 0,\ -(n\pi)^2$$

$n = 1, 2, 3, \ldots$

$H_0(y) = 1$

$H_n(y) = \cos(n\pi y)$

From b.c. (I), (II), only $c = 0$, $-9\pi^2$ $(n=3)$ matter.

$$c = 0 \Rightarrow -\frac{9}{16}\frac{G_0''}{G_0} = 0 \Rightarrow G_0'' = 0 \Rightarrow G_0(x) = A_0 x + B_0.$$

$$c = -9\pi^2 \Rightarrow -\frac{9}{16}\frac{G_3''}{G_3} = -9\pi^2 \Rightarrow G_3'' = 16\pi^2 G_3$$

$(n = 3)$

$$\Rightarrow G_3(x) = A_3\cosh(4\pi x) + B_3\sinh(4\pi x).$$

Full solution: $u(x,y) = A_0 x + B_0 + \left[A_3\cosh(4\pi x) + B_3\sinh(4\pi x)\right]\cdot\cos(3\pi y)$

$$\Rightarrow u_x(x,y) = A_0 + \left[4\pi A_3\sinh(4\pi x) + 4\pi B_3\cosh(4\pi x)\right]\cos(3\pi y)$$

From b.c. (I) $\Rightarrow$ $A_0 = 1$, $B_3 = 0$

From b.c. (II) $\Rightarrow$ $A_0 = 1$, $4\pi A_3\sinh(4\pi) = 1 \Rightarrow A_3 = \dfrac{1}{4\pi\sinh(4\pi)}$.

At the end, B_0 remains undetermined.

Full solution:

$$u(x,y) = x + B_0 + \frac{\cosh(4\pi x)\cos(3\pi y)}{4\pi\sinh(4\pi)}$$

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$\hookrightarrow$ B_0 is an arbitrary constant

$\Rightarrow$ infinite many solutions.

Q3

By seperation of variables, $u \sim G(x)H(y)$,

$$\Rightarrow \quad -\frac{\ddot{H}-4\dot{H}+7H}{H} = \boxed{\frac{G''}{G} = c \qquad \begin{array}{l} G(0)=0 \\ G(\pi)=0 \end{array}} \Rightarrow \begin{array}{l} c_n = -n^2 \\ \quad n=1,2,3,\ldots \\ G_n(x) = \sin(nx). \end{array}$$

From b.c. (iii), (iv),
only $n=2$ $(c=-4)$ matters.

$$n=2: \quad -\frac{\ddot{H}_2-4\dot{H}_2+7H_2}{H_2} = -4$$

$$\Rightarrow \ddot{H}_2 - 4\dot{H}_2 + 3H_2 = 0 \qquad \text{Let } H_2(y) \sim e^{\alpha y}$$

$$\Rightarrow \alpha^2 - 4\alpha + 3 = 0 \Rightarrow \alpha = 1, 3$$

$$\Rightarrow H_2(y) = Ae^{y} + Be^{3y}$$

Full solution: $u(x,y) = (Ae^{y} + Be^{3y})\sin(2x)$

$$\Rightarrow u_y(x,y) = (Ae^{y} + 3Be^{3y})\sin(2x)$$

From b.c. (iii) $\Rightarrow$ $A + 3B = 2$ —— ①

From b.c. (iv) $\Rightarrow$ $Ae^{\ln 2} + Be^{3\ln 2} = 2$

$$\Rightarrow 2A + 8B = 2 \quad \text{—— ②}$$

Solve ①, ② $\Rightarrow A = 5,\ B = -1$

Full solution: $u(x,y) = (5e^{y} - e^{3y})\sin(2x)$

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