MAE/MSE 502, Spring 2023 Homework #3 (14 points)

A statement of collaboration is required.

For Problem 1 and 2, the solution can be expressed as an infinite series. To compute the values of *u* for the plots, the series needs to be truncated to a finite number of terms (see relevant remarks below HW1-Q1). Matlab can be used to evaluate the expansion coefficients by numerical integration, for example using the "trapz" function.

Problem 1 (2.5 points)
For $u(x, t)$ defined on the domain of $0 \le x \le 5$ and $t \ge 0$, consider the 1-D Wave equation,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} ,$$

with the boundary conditions,

(i) $u(0, t) = 0$, (ii) $u(5, t) = 0$, (iii) $u(x, 0) = P(x)$, (iv) $u_t(x, 0) = 0$.

(a) Solve the system with $P(x)$ given as

$$P(x) = \begin{cases} x \quad , & if\ 0 \le x < 1 \\ 1.25 - 0.25\, x\, , & if\ 1 \le x \le 5 \end{cases}$$

and plot the solution as a function of x at t = 0, 1.5, 2.5, 3.5, 5, and 9. Please collect all 6 curves in one plot.

(b) Solve the system with $P(x)$ given as (this emulates a "wave packet")

$$P(x) = \sum_{n=32}^{48} \exp\left[-\left(\frac{n-40}{5}\right)^2\right] \sin\left(\frac{n\pi x}{5}\right) ,$$

and plot the solution as a function of x at t = 0, 2.5, 5, and 6. For this part, it is recommended that the four curves be plotted separately, for example arranged in 4 panels using "subplot" in Matlab.

Problem 2 (4 points)
For $u(x,t)$ defined on the domain of $0 \le x \le 2\pi$ and $t \ge 0$, consider the PDE (in which *U*, *K*, *B*, and *D* are constants)

$$\frac{\partial u}{\partial t} = U\frac{\partial u}{\partial x} + K\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^3 u}{\partial x^3} + D\frac{\partial^5 u}{\partial x^5}$$

with periodic boundary conditions in the *x*-direction. The boundary condition in the *t*-direction is given as

$$u(x, 0) = \frac{[1-\cos(x)]^8}{256} .$$

To prepare for the major tasks below, first solve the PDE by Fourier series expansion.

(a) Plot the solution $u(x, t)$ at $t = 0.2$ for the three cases: **(i)** ($U = -10, K = 0, B = 0, D = 0$), **(ii)** ($U = 0, K = 1, B = 0, D = 0$), **(iii)** ($U = -10, K = 1, B = 0, D = 0$). Also, plot the solution at $t = 0$ (which is the same for all three cases). Collect all four curves in one plot.

(b) Plot the solution $u(x, t)$ at $t = 0.2$ for the two cases: **(i)** ($U = 0, K = 0, B = 0.15, D = 0$), **(ii)** ($U = 0, K = 0, B = 0, D = 0.008$). Also, plot the solution at $t = 0$ (which is the same for the two cases). Collect all three curves in one plot.

For Problem 3-5, we expect an exact solution which consists of only a finite number of terms and no unevaluated integrals. The solution should be expressed in real numbers and functions. Expect a deduction if an imaginary number "i" ($=\sqrt{-1}$) is left in the solution.

Problem 3 (2.5 points)
For $u(x, t)$ defined on the domain of $0 \le x \le 2\pi$ and $t \ge 0$, solve the PDE,

$$\frac{\partial u}{\partial t} = 9t\frac{\partial^3 u}{\partial x^3} + \frac{\partial^4 u}{\partial x^4} + t\frac{\partial^5 u}{\partial x^5} - u \ ,$$

with periodic boundary conditions in the x-direction, and the boundary condition in the t-direction given as

$u(x, 0) = 1 + \sin(x) + \cos(3x)$.

Problem 4 (2.5 points)
For $u(x,t)$ defined on the domain of $0 \le x \le 2\pi$ and $t \ge 0$, solve the PDE

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x\, \partial t} + u \ ,$$

with periodic boundary conditions in the x-direction, and the boundary conditions at $t = 0$ given as

(i) $u(x, 0) = 1$ (ii) $u_t(x, 0) = \cos(x)$.

Problem 5 (2.5 points)
For $u(x,t)$ defined on the domain of $0 \le x \le 2\pi$ and $t \ge 0$, solve the PDE

$$\frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t} - \frac{\partial^4 u}{\partial x^4} + u = 0 \ ,$$

with periodic boundary conditions in the x-direction, and the boundary conditions at $t = 0$ given as

(i) $u(x, 0) = \sin(x)$ (ii) $u_t(x, 0) = \cos(2x)$.