Q3

By Fourier series expansion:

$$u(x,t) = \sum_{n=-\infty}^{\infty} C_n(t) e^{inx}$$

$$\Rightarrow \dot{C}_n = \left(-i9tn^3 + n^4 + itn^5 - 1\right) C_n$$

observing the b.c., only need to process $n=0$, 1, and 3:

$n=0$: $\dot{C}_0 = -C_0 \Rightarrow C_0(t) = C_0(0)e^{-t} = e^{-t}$

$n=1$: $\dot{C}_1 = -i8t\,C_1$

$$\Rightarrow C_1(t) = C_1(0) e^{-i4t^2} = \frac{1}{2i} e^{-i4t^2}$$

$$u(x,0) = \sum_{n=-\infty}^{\infty} C_n(0) e^{inx}$$

$$\| $$

$$1 + \sin(x) + \cos(3x)$$

By visual inspection:

$C_0(0) = 1$

$C_1(0) = \frac{1}{2i}$

$C_3(0) = \frac{1}{2}$

(also $C_{-1}(0) = \frac{-1}{2i}$, $C_{-3}(0) = \frac{1}{2}$)

All other $C_n(0) = 0$

$n=3$: $\dot{C}_3 = 80C_3 \Rightarrow C_3(t) = C_3(0)e^{80t} = \frac{1}{2}e^{80t}$

Full solution:

$$u(x,t) = C_0(t) + \left[\left(C_1(t)e^{ix} + C_3(t)e^{i3x}\right) + c.c.\right]$$

$$= e^{-t} + \left[\left(\frac{1}{2i} e^{-i4t^2} e^{ix} + \frac{1}{2}e^{80t}e^{i3x}\right) + c.c.\right]$$

$$= e^{-t} + 2\cdot \mathrm{Re}\left[\frac{1}{2i} e^{i(x-4t^2)} + \frac{1}{2}e^{80t}e^{i3x}\right]$$

$$= e^{-t} + \sin(x-4t^2) + e^{80t}\cos(3x)$$

#

Q4

By Fourier series expansion:

$$u(x,t) = \sum_{n=-\infty}^{\infty} C_n(t) e^{inx}$$

$$\Rightarrow \ddot{C}_n = (-n^2+1) C_n + in\dot{C}_n$$

observing the b.c.'s. only need to process $n=0, 1$:

$n=0$: $\ddot{C}_0 = C_0$

$\Rightarrow C_0(t) = A_0 \cosh(t) + B_0 \sinh(t)$

$\to \dot{C}_0(t) = A_0 \sinh(t) + B_0 \cosh(t)$

Using the information from b.c. (i), (ii):

$C_0(0) = 1 \Rightarrow A_0 = 1$, $\dot{C}_0(0) = 0 \Rightarrow B_0 = 0$

$\Rightarrow C_0(t) = \cosh(t)$.

From b.c. (i):

$$\underbrace{u(x,0)}_{=1} = \sum_{n=-\infty}^{\infty} C_n(0) e^{inx}$$

$\Rightarrow C_0(0) = 1$, all other $C_n(0) = 0$

From b.c. (ii):

$$\underbrace{u_t(x,0)}_{=\cos(x)} = \sum_{n=-\infty}^{\infty} \dot{C}_n(0) e^{inx}$$

$\Rightarrow \dot{C}_1(0) = \frac{1}{2}$ (also $\dot{C}_{-1}(0) = \frac{1}{2}$)

All other $\dot{C}_n(0) = 0$

$n=1$: $\ddot{C}_1 = i\dot{C}_1$ Let $D(t) \equiv \dot{C}_1(t) \Rightarrow \dot{D} = iD \Rightarrow D(t) = D(0)e^{it}$

But $D(0) = \dot{C}_1(0) = \frac{1}{2} \Rightarrow \dot{C}_1(t) = D(t) = \frac{1}{2}e^{it}$

$$\Rightarrow C_1(t) = \underbrace{C_1(0)}_{=0} + \frac{1}{2}\int_0^t e^{it}\,dt = \frac{1}{2i}(e^{it}-1)$$

Full solution:

$$\begin{aligned} u(x,t) &= C_0(t) + \left[C_1(t)e^{ix} + \text{c.c.}\right] \\ &= \cosh(t) + \left[\left(\frac{1}{2i}e^{i(x+t)} - \frac{1}{2i}e^{ix}\right) + \text{c.c.}\right] \\ &= \cosh(t) + 2\cdot \mathrm{Re}\left[\frac{1}{2i}e^{i(x+t)} - \frac{1}{2i}e^{ix}\right] \\ &= \cosh(t) + \sin(x+t) - \sin(x) \end{aligned}$$

#

Q5

By Fourier series expansion:

$$u(x,t) = \sum_{n=-\infty}^{\infty} C_n(t)\, e^{inx}$$

$$\Rightarrow \ddot{C}_n - 2\dot{C}_n + (1-n^4)\, C_n = 0$$

Observing the b.c.'s, only need to process $n = 1, 2$:

$n=1$: $\ddot{C}_1 = 2\dot{C}_1$ Let $D(t) \equiv \dot{C}_1(t)$

$$\Rightarrow \dot{D} = 2D, \quad D(t) = D(0)\, e^{2t} = \dot{C}_1(0)\, e^{2t} = 0$$

because $\dot{C}_1(0) = 0$

$$\dot{C}_1 = 0 \Rightarrow C_1(t) = C_1(0) = \frac{1}{2i}$$

From b.c. (i):

$$u(x,0) = \sum_{n=-\infty}^{\infty} C_n(0)\, e^{inx}$$

$$\sin(x) = u(x,0)$$

$$\Rightarrow C_1(0) = \frac{1}{2i} \quad \left(\text{also } C_{-1}(0) = \frac{-1}{2i}\right)$$

All other $C_n(0) = 0$.

From b.c. (ii):

$$u_t(x,0) = \sum_{n=-\infty}^{\infty} \dot{C}_n(0)\, e^{inx}$$

$$\cos(2x) = u_t(x,0)$$

$$\Rightarrow \dot{C}_2(0) = \frac{1}{2} \quad \left(\text{also } \dot{C}_{-2}(0) = \frac{1}{2}\right)$$

All other $\dot{C}_n(0) = 0$.

$n=2$:

$$\ddot{C}_2 - 2\dot{C}_2 - 15 C_2 = 0, \quad \text{Let } C_2(t) \sim e^{\alpha t}$$

$$\Rightarrow \alpha^2 - 2\alpha - 15 = 0 \Rightarrow \alpha = 5, -3 \Rightarrow C_2(t) = A_2 e^{5t} + B_2 e^{-3t}$$

$$\Rightarrow \dot{C}_2(t) = 5A_2 e^{5t} - 3B_2 e^{-3t}$$

From b.c.'s:

$$\left.\begin{aligned} C_2(0) = 0 &\Rightarrow A_2 + B_2 = 0 \\ \dot{C}_2(0) = \tfrac{1}{2} &\Rightarrow 5A_2 - 3B_2 = \tfrac{1}{2} \end{aligned}\right\} \text{solve} \Rightarrow A_2 = \frac{1}{16},\ B_2 = -\frac{1}{16}$$

$$\Rightarrow C_2(t) = \frac{1}{16}\left(e^{3t} - e^{-t}\right)$$

Full solution:

$$u(x,t) = \left(C_1(t)\, e^{ix} + C_2(t)\, e^{i2x}\right) + \text{c.c.}$$

$$= \left[\frac{1}{2i} e^{ix} + \frac{1}{16}\left(e^{5t} - e^{-3t}\right) e^{i2x}\right] + \text{c.c.}$$

$$= 2 \cdot \mathrm{Re}\left[\frac{1}{2i} e^{ix} + \frac{1}{8}\left(e^{5t} - e^{-3t}\right) e^{i2x}\right] + \text{c.c.}$$

$$= \sin(x) + \frac{1}{8}\left(e^{5t} - e^{-3t}\right)\cos(2x)$$

#