Q1

From the homogeneous subsystem, determine $\{\sin(nx)\}_{n=1,2,3,\dots}$ as b.b.

$$\Rightarrow \quad u(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin(nx) \quad \text{— ①}$$

$$Q(x,t) = \sum_{n=1}^{\infty} g_n(t)\sin(nx) \quad \text{— ②}$$

$$Q(x,t) = e^{-2t}\sin(x) \Rightarrow g_1(t) = e^{-2t} \text{ all other } g_n(t)=0.$$

From b.c. (iii),

$a_2(0) = 1$

All other $a_n(0)=0$

plugging ①, ② into PDE:

$$\frac{da_n}{dt} = (-3n^2+1)a_n + g_n$$

Only $n=1,2$ matter.

$$n=1: \quad \frac{da_1}{dt} = -2a_1 + e^{-2t} \Rightarrow a_1(t) = \underbrace{a_1(0)}_{=0}e^{-2t} + \int_0^t e^{-2\hat{t}}e^{-2(t-\hat{t})}\,d\hat{t}$$

$$= t e^{-2t}$$

$$n=2: \quad \frac{da_2}{dt} = -11a_2 \Rightarrow a_2(t) = \underbrace{a_2(0)}_{=1}e^{-11t} = e^{-11t}$$

Full solution:

$$u(x,t) = a_1(t)\sin(x) + a_2(t)\sin(2x)$$

$$= te^{-2t}\sin(x) + e^{-11t}\sin(2x)$$

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Q2

By Fourier series expansion,

$$u(x,t) = \sum_{n=-\infty}^{\infty} C_n(t) e^{inx} \quad \text{—— ①}$$

$$Q(x,t) = \sum_{n=-\infty}^{\infty} q_n(t) e^{inx} \quad \text{—— ②}$$

$$Q(x,t) = \cos(x)\cos(t) = \frac{e^{ix}+e^{-ix}}{2}\cdot\cos(t) \longrightarrow q_1(t) = \frac{1}{2}\cos(t)$$

All other $q_n = 0$

$\left(\text{also, } q_{-1}(t) = \frac{1}{2}\cos(t)\right)$

From the b.c.'s:

$$u(x,0) = \sum_{n=-\infty}^{\infty} C_n(0) e^{inx}$$

$$u(x,0) = \sin(x) = \frac{e^{ix}-e^{-ix}}{2i} \rightarrow C_1(0) = \frac{1}{2i}$$

$\left(\text{also } C_{-1}(0) = \frac{-1}{2i}\right)$

All other $C_n(0) = 0$

$$u_t(x,0) = \sum_{n=-\infty}^{\infty} \dot{C}_n(0) e^{inx}$$

$$u_t(x,0) = 1 \rightarrow \dot{C}_0(0) = 1$$

All other $\dot{C}_n(0) = 0$.

Plugging ①, ② into PDE:

$$\ddot{C}_n = (n^4-1) C_n + q_n$$

Only need to process $n=0, 1$

$n=0$: $\ddot{C}_0 = -C_0$

$$C_0(t) = A_0\cos(t) + B_0\sin(t)$$

$$\Rightarrow \dot{C}_0(t) = -A_0\sin(t) + B_0\cos(t)$$

From b.c.'s: $C_0(0)=0 \Rightarrow A_0 = 0$, $\dot{C}_0(0) = 1 \Rightarrow B_0 = 1$

$$\Rightarrow C_0(t) = \sin(t)$$

$n=1$: $\ddot{C}_1 = q_1 = \frac{1}{2}\cos(t) \Rightarrow \dot{C}_1(t) = \overbrace{\dot{C}_1(0)}^{0} + \frac{1}{2}\sin(t) = \frac{1}{2}\sin(t)$

$$C_1(t) = C_1(0) + \frac{1}{2}(1-\cos t)$$

$$= \frac{1}{2i} + \frac{1}{2}(1-\cos t)$$

Full solution: $u(x,t) = C_0(t) + \left(C_1(t)e^{ix} + \text{c.c.}\right)$

$$= \sin(t) + \left\{\left[\frac{1}{2i} + \frac{1}{2}(1-\cos(t))\right]e^{ix} + \text{c.c.}\right\}$$

$$= \sin(t) + 2\cdot \mathrm{Re}\left\{\left[\frac{1}{2i} + \frac{1}{2}(1-\cos(t))\right]e^{ix}\right\}$$

$$= \sin(t) + \sin(x) + [1-\cos(t)]\cos(x)$$

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Q4

First, seek steady solution $u_s(x)$:

$$u_s'' = -4 + \pi^2 \sin(\pi x), \quad \begin{array}{|l} u_s(0) = 1 \quad \text{— (i)} \\ u_s(1) = -1 \quad \text{— (ii)} \end{array}$$

$$\Rightarrow u_s' = -4x - \pi\cos(\pi x) + A$$

$$\Rightarrow u_s(x) = -2x^2 - \sin(\pi x) + Ax + B$$

From b.c. (i) $\Rightarrow B = 1$ From b.c. (ii) $\Rightarrow A = 0$

$$\Rightarrow u_s(x) = 1 - 2x^2 - \sin(\pi x)$$

Let $\hat{u}(x,t) \equiv u(x,t) - u_s(x)$

$$\Rightarrow (\maltese) \begin{cases} \dfrac{\partial \hat{u}}{\partial t} = \dfrac{\partial^2 \hat{u}}{\partial x^2} \\ \hat{u}(0,t) = 0 \\ \hat{u}(1,t) = 0 \\ \hat{u}(x,0) = (1-2x^2) - u_s = \sin(\pi x) \end{cases}$$

Solving $(\maltese)$, $\hat{u}(x,t) = \sin(\pi x)\, e^{-\pi^2 t}$

$$\Rightarrow \text{Full solution } u(x,t) = \hat{u}(x,t) + u_s(x)$$
$$= \sin(\pi x)\left(e^{-\pi^2 t} - 1\right) + 1 - 2x^2$$

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