Q1

By MOC: $\frac{dx}{dt} = 3u + e^{3t}$

$$\frac{du}{dt} = 3x$$

Let $v \equiv x + u \Rightarrow \frac{dv}{dt} = 3v + e^{3t}$

$$\Rightarrow v(t) = v(0)e^{3t} + te^{3t}$$

$$\Rightarrow u(t) + x(t) = [u(0) + x(0)]e^{3t} + te^{3t}$$

From b.c., $u(0) = 1 - x(0)$

$$\Rightarrow u(t) + x(t) = (1+t)e^{3t}$$

$$u(x(t), t) = -x(t) + (1+t)e^{3t}$$

Full solution: $u(x,t) = -x + (1+t)e^{3t}$

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Q2

By MOC: $\dfrac{dx}{dt} = e^{u} - t$ — ①

$\dfrac{du}{dt} = e^{-u}$ — ②

From ②: $e^{u}du = dt \Rightarrow e^{u(t)} - e^{u(0)} = t$ — ③

$\Rightarrow u(t) = \ln(e^{u(0)} + t)$ — ④

Plugging ③ into ①:

$$\frac{dx}{dt} = (e^{u(0)} + t) - t = e^{u(0)}$$

$$\Rightarrow x(t) = x(0) + e^{u(0)} \cdot t \quad \text{— ⑤}$$

If $x(0) < 1 \Rightarrow u(0) = 0 \xRightarrow{④} \boxed{u(t) = \ln(1+t)}$

$\overset{⑤}{\Downarrow}$ $x(t) = x(0) + t$

$\boxed{x(t) < 1 + t}$

If $x(0) \geq 1 \Rightarrow u(0) = \ln(x(0)) \xRightarrow{④} u(t) = \ln(x(0) + t)$

$\overset{⑤}{\Downarrow}$ $x(t) = (1+t)\, x(0)$

$\boxed{x(t) \geq 1 + t}$

$\boxed{u(t) = \ln\left(\dfrac{x(t)}{1+t} + t\right)}$

Full solution:

$$u(x,t) = \begin{cases} \ln(1+t), & \text{if } x < 1+t \\ \ln\left(\dfrac{x}{1+t} + t\right), & \text{if } x \geq 1+t \end{cases}$$

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Q3

By MOC: $\dfrac{dx}{dt} = x$ — ①

$\dfrac{du}{dt} = x u e^{-t} + 1$ — ②

From ①: $x(t) = x(0) e^{t} \Rightarrow x(0) = x(t) e^{-t}$

$$\Rightarrow \frac{du}{dt} = x(0) e^{t} \cdot u e^{-t} + 1 = x(0) u + 1 \quad — ③$$

Solution of ③ is $u(t) = u(0) e^{x(0)t} + e^{x(0)t} \int_0^t e^{-x(0)\hat{t}} d\hat{t}$

From b.c., $u(0) = 1$ $\quad = e^{x(0)t} + \dfrac{1}{x(0)} (e^{x(0)t} - 1)$ — ④

$$\Rightarrow u(t) = e^{x(t)e^{-t} \cdot t} + \frac{1}{x(t) e^{-t}} [e^{x(t)e^{-t} \cdot t} - 1]$$

Full solution:

$$u(x,t) = e^{xe^{-t} \cdot t} + \frac{1}{xe^{-t}} [e^{xe^{-t} \cdot t} - 1]$$

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Note: The solution is not singular at $x = 0$. In fact, at $x = 0$, $u(x,t) = 1 + t$. This can be obtained by applying L'Hospital's rule to the "full solution" above. Or, even simpler, observe that when $x(t) = 0 \Leftrightarrow x(0) = 0$, the solution of ③ is reduced to $u(t) = u(0) + t = 1 + t$.

Q4

First, write the PDE as

$$\underbrace{\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) u}_{} = x$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\underbrace{\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) u}_{w}$$

$$\Rightarrow \begin{cases} \dfrac{\partial w}{\partial t} - \dfrac{\partial w}{\partial x} = x & \text{— ①} \\[2ex] \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} = w & \text{— ②} \end{cases}$$

Since

$$w(x,t) = u_t(x,t) + u_x(x,t),$$

$$w(x,0) = u_t(x,0) + u_x(x,0)$$
$$= 1 + 1 = \underline{\underline{2}} \quad \text{— (iii)}$$

Solve ① by MOC with b.c. (iii) :

$$\frac{dx}{dt} = -1 \Rightarrow x(t) = x(0) - t \Rightarrow x(0) = x(t) + t$$

$$\frac{dw}{dt} = x = x(0) - t \Rightarrow w(t) = w(0) + x(0)t - \frac{t^2}{2}$$
$$= 2 + (x(t)+t)\cdot t - \frac{t^2}{2}$$

$$\Rightarrow w(x,t) = 2 + xt + \frac{t^2}{2}$$

Solve ② by MOC with b.c. (i)

$$\frac{dx}{dt} = 1 \Rightarrow x(t) = x(0) + t \Rightarrow x(0) = x(t) - t$$

$$\frac{du}{dt} = w = 2 + xt + \frac{t^2}{2} = 2 + (x(0)+t)\cdot t + \frac{t^2}{2}$$

$$\Rightarrow u(t) = u(0) + 2t + x(0)\frac{t^2}{2} + \frac{t^3}{2}$$

From b.c. (i), $u(0) = x(0)$

$$= x(0) + 2t + x(0)\cdot\frac{t^2}{2} + \frac{t^3}{2}$$
$$= (x(t) - t)\left(1 + \frac{t^2}{2}\right) + 2t + \frac{t^3}{2}$$
$$= x(t) + \frac{x(t)t^2}{2} + t$$

Full solution: $u(x,t) = x + \dfrac{xt^2}{2} + t$

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