MAE502 Homework #1 (1 point = 1% of your total score for the semester)

Prob. 1 (5 points)

(a) Solve the heat equation for $u(x, t)$, $x \in [0,1]$, $t \in [0, \infty)$,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} ,$$

with the boundary conditions,

(i) $u(0,t) = 0$ (ii) $u_x(1,t) = 0$ ($u_x \equiv \partial u/\partial x$) (iii) $u(x,0) = 2\sin(\frac{\pi x}{2}) + \sin(\frac{9\pi x}{2})$

[This mathematical model, with u as temperature, describes heat transfer along a metal rod, one end of it ($x = 1$) is insulated with zero heat flux while the other end ($x = 0$) is attached to a thermal reservoir to keep its temperature fixed at $u = 0$. Note that $\phi \equiv -\partial u/\partial x$ is heat flux. Boundary condition (iii) provides the initial distribution of temperature.]

(b) Find the equilibrium solution, $u(x,t)$ as $t \to \infty$.

(c) From (a) and (b), plot the solution $u(x,t)$ at $t = 0$ (initial state), 0.003, 0.01, 0.1, 0.3, 1, and $t \to \infty$. (Please collect them in a single plot.) Interpret your result.

(d) From (a) and (b), find the analytic expression of heat flux, $\phi \equiv -\partial u/\partial x$. Plot $\phi(x,t)$ at $t = 0$, 0.003, 0.01, 0.1, 0.3, 1, and $t \to \infty$. Interpret your result.

(e) At $x = 0$, temperature is fixed but heat flux is allowed to change. Plot the heat flux at $x = 0$, $\phi(0,t)$, as a function of t. Does heat energy flow into or out of the metal rod through this end?

(f) The parameter, $S(t) \equiv \int_0^1 (\frac{\partial u}{\partial x})^2 dx$, is a measure of the sharpness of temperature gradient (a smoother temperature profile corresponds to a smaller S). Derive the analytic expression for $S(t)$ and plot it as a function of t. Discuss the result.

Prob. 2 (3 points)

Find the general solution of each of the following PDEs by the method of *separation of variables*.

(a) $\frac{\partial^2 u}{\partial x \partial y} + u = 0$ (b) $\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 0$