MAE 502 Homework #4

Prob 1 (5 points)

(a) Solve the nonhomogeneous PDE (heat equation with internal heat source/sink),

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + Q(x,t) \quad ,$$

for $u(x, t)$ defined on $x \in [0,1]$, $t \in [0, \infty)$, with the boundary conditions

(I) $u(0, t) = 0$, (II) $u(1, t) = 0$, (III) $u(x, 0) = 2\sin(\pi x) + 5\sin(2\pi x)$,

and with $Q(x, t)$ given as

$$Q(x, t) = \sin(\pi x)\exp(-2t) + \sin(3\pi x)\exp(-t).$$

(b) Using the solution in (a), find the values of $u(x,t)$ at $(x, t) = (0.75, 0.1)$ and $(0.75, 0.3)$.

Prob 2 (4 points)

(a) Given the following function defined on the semi-infinite interval, $x \in [0, \infty)$,

$$f(x) = 1 \;,\; 0 \le x \le 1, \qquad = 0 \;,\; x > 1 \;, \qquad \text{Eq. (1)}$$

determine the Fourier Sine transform of $f(x)$, $F(\omega)$, that satisfies

$$f(x) = \int_0^\infty F(\omega)\sin(\omega x)\,d\omega \quad .$$

Plot $F(\omega)$ as a function of ω for the range $0 \le \omega \le 30$.

(b) If the $f(x)$ in Eq. (1) is instead defined on a finite interval, $x \in [0, L]$, $L > 1$ (but otherwise retains its definition in Eq. (1)), find the coefficients, a_n, for the Fourier Sine series of $f(x)$,

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \quad .$$

Plot a_n as a function of n for the following cases: (i) For $L = 2$, plot a_n for the range $1 \le n < 60/\pi$. (ii) For $L = 5$, plot a_n for $1 \le n < 150/\pi$. (iii) For $L = 100$, plot a_n for $1 \le n < 3000/\pi$. Compare these plots with the plot of $F(\omega)$ in (a). Discuss your results. (*Note: This homework illustrates the correspondence between Fourier series and Fourier integral.*)