Introduction to Partial Differential Equation - II. Some simple analytic solutions

We will restrict the discussion here to the method of *separation of variables*. Let's start with some simple examples of the general solutions of PDFs without invoking boundary conditions.

Example 1: Solve $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$

Step 1. Assume that $u(x,y) = G(x)H(y)$, i.e., u can be written as the product of two functions, one depends only on x, the other depends only on y. This leads to

$$H\frac{dG}{dx} + G\frac{dH}{dy} = 0 \quad . \tag{1}$$

Step 2. Rearrange the equation to collect all that depend only on x in one side, and all that depend only on y in the other. This can be done by multiplying Eq. (1) by 1/(GH), then rearrange it into

$$\frac{1}{G}\frac{dG}{dx} = \frac{-1}{H}\frac{dH}{dy} \quad . \tag{2}$$

Step 3. But, if two functions, $f(x)$ that depends only on x and $g(y)$ that depends only on y, equal each other for all x and y, there can be only one possibility, namely, $f(x) = g(y) =$ *a constant*. Therefore, Eq. (2) implies

$$\frac{1}{G}\frac{dG}{dx} = \frac{-1}{H}\frac{dH}{dy} = c \quad ,$$

where c is a constant. Thus, we are splitting the original PDE into a pair of ODEs that are connected by a common constant,

$$\frac{1}{G}\frac{dG}{dx} = c \quad , \tag{3}$$

$$\frac{1}{H}\frac{dH}{dy} = -c \quad . \tag{4}$$

The general solutions for (3) and (4) are G(x) = k_1 exp(cx) and H(y) = k_2 exp(−cy), where k_1 and k_2 are just two arbitrary constants that can be combined when we multiply G to H to reconstruct u. This leads to

$$u(x, y) = k \exp(cx - cy) \quad ,$$

where k is an arbitrary constant. It can be readily verified that this solution satisfies the original PDE.

(Although the procedure here looks simple, more complicated situations will emerge once boundary conditions get involved. This will be discussed shortly.)

Example 2: Solve $y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 0$

Again, assume u(x, y) = G(x)H(y) so the PDE is transformed into

$$yH\frac{dG}{dx} - xG\frac{dH}{dy} = 0 \quad .$$

Following the argument in Ex. 1, the above equation implies

$$\frac{1}{xG}\frac{dG}{dx} = \frac{1}{yH}\frac{dH}{dy} = c \quad ,$$

or

$$\frac{1}{xG}\frac{dG}{dx} = c \quad , \qquad (5)$$

$$\frac{1}{yH}\frac{dH}{dy} = c \quad . \qquad (6)$$

From (5) and (6) we obtain G(x) = k_1 exp(c x^2/2) and H(y) = k_2 exp(c y^2/2), therefore our solution is

$$u(x, y) = k \exp[(c/2)(x^2 + y^2)] \quad .$$

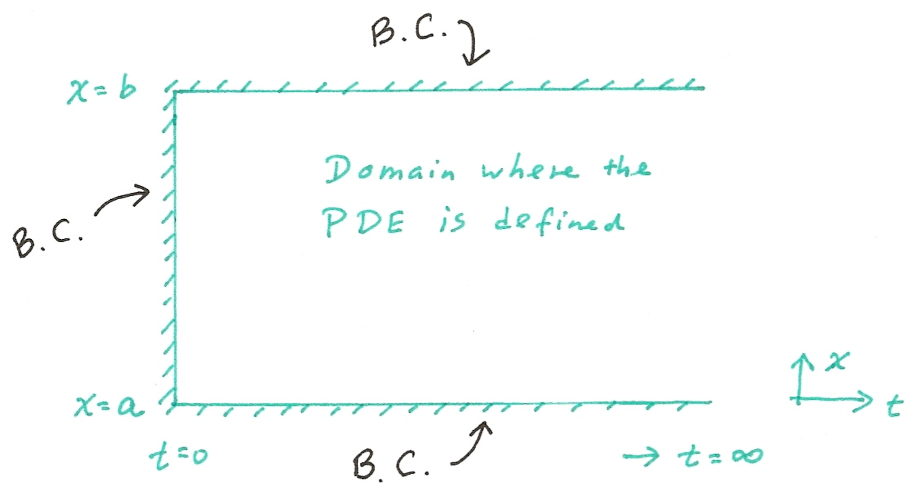
When boundary conditions are introduced to the problem, the approach of separation of variables usually leads to an eigenvalue problem

Example 3: For $u(x, t)$ defined on $x \in [0, 1]$ and $t \in [0, \infty)$, solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} ,$$

with boundary conditions (I) $u(0, t) = 0$, (II) $u(1, t) = 0$,and (III) $u(x, 0) = \sin(4\pi x)$

This is the heat equation defined on a bounded interval in x and semi-infinite domain in t, as mentioned in Part I. Let's repeat the relevant diagram from Part I (with $a = 0$ and $b = 1$):



Here, the three boundary conditions (I), (II), and (III) have to be satisfied at the three "walls" at bottom, top, and left, respectively.

(Ex. 3 continued)

Step 1: Separation of variables. Let $u(x, t) = \mathrm{G}(x)\mathrm{H}(t)$, the usual procedure leads to

$$\frac{1}{G}\frac{d^2 G}{d x^2} = \frac{1}{H}\frac{d H}{d y} = c \quad ,$$

or

$$\frac{1}{G}\frac{d^2 G}{d x^2} = c \quad , \tag{7}$$

$$\frac{1}{H}\frac{d H}{d y} = c \quad . \tag{8}$$

Since the boundary conditions (I) and (II) have to be satisfied for all t, they are reduced to

(IV) G(0) = 0 , (V) G(1) = 0 .

We will see that in order for the solution of Eq. (7) to satisfy (IV) and (V) and be non-trivial (i.e., G(x) is not identically zero), the "constant" c must be a certain specific values. **Equation (7) plus the boundary conditions (IV) and (V) form an eigenvalue problem**.

So far, we have not determined whether c should be positive or negative (we will see that this matters). The only information we gain from separation of variable is that c is a common constant for Eqs. (7) and (8). In other words, if it is determined from the eigenvalue problem of (7)+(IV)+(V) that c equals a specific value (one of the eigenvalues), this same value must be used for the "c" in Eq. (8) before we proceed to solve (8) and combine G(x) and H(t) to reconstruct $u(x, t)$.

(Ex. 3 continued)

Step 2: Solve the eigenvalue problem for the eigenvalue c and eigenfunction G(x).

It can be readily shown that if $c > 0$ the only solution of (7)+(IV)+(V) is a trivial solution, $G(x) \equiv 0$. (Exercise: Verify it. Consult your ODE textbook.) We then try the possibility of $c < 0$. In this case, it is convenient to rewrite c as

$$c = -k^2 .$$

With this setting, the general solution of Eq. (7) is

$$G(x) = A\, cos(k\, x) + B\, sin(k\, x) .$$

From boundary condition (IV), $A = 0$. From b. c. (V), and demanding that $G(x)$ be non-trivial, we have

$$sin(k\, x) = 0 .$$

The only values of k that satisfy this condition are $k_1 = \pi$, $k_2 = 2\pi$, ..., $k_N = N\pi$, ... Thus, our eigenvalues are

$$c_1 = -\pi^2,\ \ c_2 = -4\,\pi^2,\ \ c_3 = -9\,\pi^2 ,\ ...\ c_N = -N^2\,\pi^2 ,...$$

and the corresponding eigenfunctions are

$$G_1(x) = sin(\pi x),\ \ G_2(x) = sin(2\pi x),\ G_3(x) = sin(3\pi x),\ ...\ G_N(x) = sin(N\pi x),\ ... \qquad (9)$$

(Ex. 3 continued)

Step 3: For a given eigenvalue $c = c_N$, solve the other ODE for $H(t)$. This leads to

$$H_N(t) = exp(-c_N t) = \exp(-N^2 \pi^2 t) \tag{10}$$

Step 4: Combine $G_N(x)$ and $H_N(t)$ to form the full eigenfunction,

$$u_N(x, t) = G_N(x)\, H_N(t) = sin(N \pi x) \exp(-N^2 \pi^2 t) \,. \tag{11}$$

Here, the subscript "N" indicates the N-th eigenfunction.

Step 5: Represent the full solution as the linear combination of all eigenfunctions.

Since each of the eigenfunctions satisfy the PDE and the boundary conditions, and since the PDE is linear, the full solution, u(x, t), can be represented by the linear combination of all eigenfunctions, i.e.,

$$u(x, t) = a_1\, u_1(x, t) + a_2\, u_2(x, t) + a_3\, u_3(x, t) + \dots \;,$$

or

$$u(x,t) = \sum_{n=1}^{\infty} a_n\, u_n(x,t) \quad . \tag{12}$$

Here, the coefficients, a_n, are yet to be determined.

(Ex. 3 continued)

Step 6: Use the boundary condition (III) at $t = 0$ to determine the coefficients, a_n ,in Eq. (12).

From Eqs. (11) and (12), we have

$$u(x,0) = \sum_{n=1}^{\infty} a_n\, u_n(x,0) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) \quad . \tag{13}$$

In order for this to satisfy b.c. (III), $u(x, 0) = sin(4\pi x)$, we must have

$a_4 = 1$, and $a_n = 0$ for all $n \neq 4$.

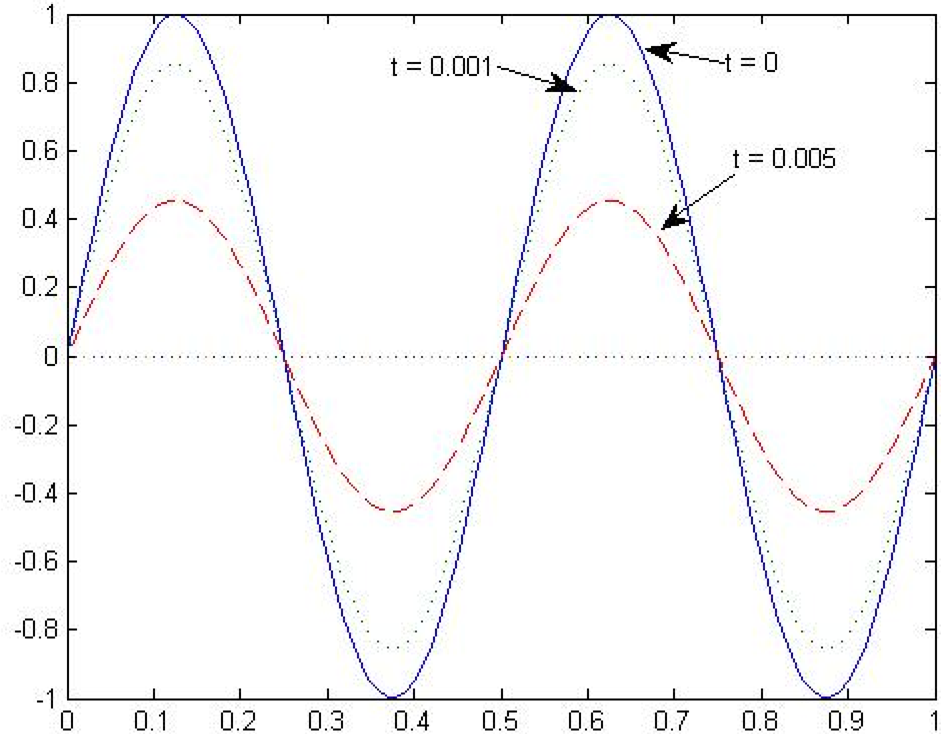
Using this last piece of information and Eqs. (11) and (12), we obtain the final solution as

$$u(x, t) = \sin(4\pi x) \exp(-16\pi^2 t) \quad .$$

It satisfies the PDE and all three boundary conditions. The figure in the next page is a plot for $u(x, t)$ at $t = 0$, $t = 0.001$, and $t = 0.005$.

Example 3 is about the simplest case in which an analytic solution can be constructed for a physically meaningful PDE with full boundary conditions. The subject is certainly more complicated than ODE. This short discussion only provides a flavor of it.

Most of the PDEs in practical applications are solved numerically. We will discuss numerical solutions in Part 3.



t = 0.001
t = 0
t = 0.005
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1