Solve the eigenvalue problem

\[ G''(x) = c \, G(x), \text{ with b.c.: (I) } G(0) = 0, \quad (II) \ G(1) = 0 \]

Observation: In the ODE, the second derivative of \( G \) is proportional to \( G \) itself. Two types of functions possess this property:

(i) \( \{ \sin(x), \cos(x) \} \)

\[
\begin{align*}
[\sin(x)]' &= \cos(x), \\
[\sin(x)]'' &= [\cos(x)]' = -\sin(x) \quad \text{← Comes back to itself but with a negative sign}
\end{align*}
\]

\( \cos(x) \) behaves similarly; \( [\cos(x)]'' = -\cos(x) \)

\( \Rightarrow \) The combination of \( G = A \sin(\alpha \, x) + B \cos(\alpha \, x) \) satisfies \( G'' = -\alpha^2 \, G \)

(ii) \( \{ \exp(x), \exp(-x) \} \)

\[
\begin{align*}
[\exp(x)]' &= \exp(x), \quad [\exp(x)]'' = \exp(x) \quad \text{← Comes back to itself}
[\exp(-x)]' &= -\exp(-x), \quad [\exp(-x)]'' = \exp(-x) \quad \text{← Comes back to itself}
\end{align*}
\]

\( \Rightarrow \) The combination of \( G = A \exp(\alpha \, x) + B \exp(-\alpha \, x) \) satisfies \( G'' = \alpha^2 \, G \)
For the convenience of discussion, we often replace \( \{\exp(x), \exp(-x)\} \) by the equivalent pair of \( \{\sinh(x), \cosh(x)\} \)

Recall that
\[
\sinh(x) \equiv \frac{\exp(x) - \exp(-x)}{2}, \quad \cosh(x) \equiv \frac{\exp(x) + \exp(-x)}{2}
\]

\[
[\sinh(x)]' = \cosh(x), \quad [\sinh(x)]'' = [\cosh(x)]' = \sinh(x), \quad \text{etc.}
\]

\[
=> \text{The combination of } \ G = A \sinh(\alpha x) + B \cosh(\alpha x) \text{ satisfies } G'' = \alpha^2 G
\]

Let's get back to the eigenvalue problem. From the above observation,

(1) When \( c > 0 \), solution is \( G(x) = A \sinh(\alpha x) + B \cosh(\alpha x) \), where \( c = \alpha^2 \)

(2) When \( c < 0 \), solution is \( G(x) = A \sin(\alpha x) + B \cos(\alpha x) \), where \( c = -\alpha^2 \)

The situation with \( c = 0 \) may also be relevant. Let's delay the discussion for that case.
Case 1: \( c > 0, G(x) = A \sinh(\alpha x) + B \cosh(\alpha x) \), where \( c = \alpha^2 \)

from b.c. (I): \( A \sinh(0) + B \cosh(0) = 0 \Rightarrow B = 0 \)

from b.c. (II): \( A \sinh(\alpha) + B \cosh(\alpha) = 0 \Rightarrow A \sinh(\alpha) = 0 \Rightarrow A = 0 \) or \( \alpha = 0 \)

If \( A = 0 \), solution is trivial, \( G \equiv 0 \)

If \( \alpha = 0 \), \( G = \text{constant} \), but b.c. (I) or (II) again leads to \( G \equiv 0 \)

**Conclusion:** \( c > 0 \) only leads to a trivial solution

Case 2: \( c < 0, G(x) = A \sin(\alpha x) + B \cos(\alpha x) \), where \( c = -\alpha^2 \)

from b.c. (I): \( A \sin(0) + B \cos(0) = 0 \Rightarrow B = 0 \)

from b.c. (II): \( A \sin(\alpha) + B \cos(\alpha) = 0 \Rightarrow A \sin(\alpha) = 0 \Rightarrow A = 0 \) or \( \sin(\alpha) = 0 \)

If \( A = 0 \), solution is trivial

**Conclusion:** For \( c < 0 \), non-trivial solutions exist when \( \sin(\alpha) = 0 \)
The values of $\alpha$ that satisfy $\sin(\alpha) = 0$ are

$$\alpha_0 = 0, \; \alpha_1 = \pi, \; \alpha_2 = 2 \pi, \; \alpha_3 = 3 \pi, \ldots \; \alpha_n = n \pi, \ldots$$

The corresponding **eigenvalues** are $c_n = -\alpha_n^2$, or

$$c_0 = 0, \; c_1 = -\pi^2, \; c_2 = -4 \pi^2, \; c_3 = -9 \pi^2, \ldots, \; c_n = -n^2 \pi^2,$$

Plugging this back to the expression of $G(x)$ for the case with $c < 0$, we obtain the **eigenfunctions**

$$G_0(x) = 0 \text{ (trivial)}, \; G_1(x) = \sin(\pi x), \; G_2(x) = \sin(2\pi x), \ldots, \; G_n(x) = \sin(n\pi x), \ldots$$

Lastly, for the case with $c = 0$, the ODE is reduced to $G''(x) = 0$, whose general solution is $G(x) = Ax + B$. From b.c. (I), we have $B = 0$; From b.c. (II), we have $A = 0$, so the solution is trivial.

**Beware that for some eigenvalue problems the zero eigenvalue, $c = 0$, can correspond to a non-trivial solution. Don't dismiss the case outright!**
Exercises: Solve the following eigenvalue problems

(i) $G''(x) = c \, G(x), \ G(3) = 0, \ G(5) = 0$

(ii) $G''(x) = c \, G(x), \ G(0) = 0, \ G'(1) = 0$ \ ($G'$ is $dG/dx$)

(iii) $G''(x) = c \, G(x), \ G'(0) = 0, \ G(1) = 0$

(iv) $G''(x) = c \, G(x), \ G(0) = G(1), \ G'(0) = G'(1)$ \ (periodic boundary condition)