Solve the eigenvalue problem

G''(x) = c G(x) , with b.c. : (I) G(0) = 0, (II) G(1) = 0

Observation: In the ODE, the second derivative of G is proportional to G itself. Two types of functions possess this property:

(i) {sin(x), cos(x)}

[sin(x)]' = cos(x) ,
[sin(x)]'' = [cos(x)]' = – sin(x) ←Comes back to itself but with a negative sign

cos(x) behaves similarly; [cos(x)]'' = – cos(x)

=> The combination of $G = A\sin(\alpha x)+B\cos(\alpha x)$ satisfies $G'' = -\alpha^2 G$

(ii) {exp(x), exp(–x)}

[exp(x)]' = exp(x) , [exp(x)]'' = exp(x) ←Comes back to itself
[exp(–x)]' = – exp(–x) , [exp(–x)]'' = exp(–x) ←Comes back to itself

=> The combination of $G = A\exp(\alpha x)+B\exp(-\alpha x)$ satisfies $G'' = \alpha^2 G$

For the convenience of discussion, we often replace {exp(x), exp(–x)} by the equivalent pair of {sinh(x), cosh(x)}

Recall that $\sinh(x) \equiv \{\exp(x) - \exp(-x)\}/2$, $\cosh(x) \equiv \{\exp(x)+\exp(-x)\}/2$

$$[\sinh(x)]' = \cosh(x) \ , \ [\sinh(x)]'' = [\cosh(x)]' = \sinh(x) \ , \text{etc.}$$

=> The combination of $G = A \sinh(\alpha x)+B \cosh(\alpha x)$ satisfies $G'' = \alpha^2 G$

Let's get back to the eigenvalue problem. From the above observation,

(1) When $c > 0$, solution is $G(x) = A \sinh(\alpha x)+B \cosh(\alpha x)$, where $c = \alpha^2$

(2) When $c < 0$, solution is $G(x) = A \sin(\alpha x)+B \cos(\alpha x)$, where $c = -\alpha^2$

The situation with $c = 0$ may also be relevant. Let's delay the discussion for that case.

Case 1: $c > 0$, $G(x) = A\sinh(\alpha x) + B\cosh(\alpha x)$, where $c = \alpha^2$

from b.c. (I): $A\sinh(0) + B\cosh(0) = 0 \Rightarrow B = 0$

from b.c. (II): $A\sinh(\alpha) + B\cosh(\alpha) = 0 \Rightarrow A\sinh(\alpha) = 0 \Rightarrow A = 0$ or $\alpha = 0$

If $A = 0$, solution is trivial, $G \equiv 0$

If $\alpha = 0$, G = constant, but b.c. (I) or (II) again leads to $G \equiv 0$

Conclusion: $c > 0$ only leads to a trivial solution

Case 2: $c < 0$, $G(x) = A\sin(\alpha x) + B\cos(\alpha x)$, where $c = -\alpha^2$

from b.c. (I): $A\sin(0) + B\cos(0) = 0 \Rightarrow B = 0$

from b.c. (II): $A\sin(\alpha) + B\cos(\alpha) = 0 \Rightarrow A\sin(\alpha) = 0 \Rightarrow A = 0$ or $\sin(\alpha) = 0$

If $A = 0$, solution is trivial

Conclusion: For $c < 0$, non-trivial solutions exist when $\sin(\alpha) = 0$

The values of α that satisfy $\sin(\alpha) = 0$ are

$$\alpha_0 = 0,\ \alpha_1 = \pi,\ \alpha_2 = 2\pi,\ \alpha_3 = 3\pi, \ldots\ \alpha_n = n\pi, \ldots$$

The corresponding **eigenvalues** are $c_n = -\alpha_n^2$, or

$$c_0 = 0,\ c_1 = -\pi^2,\ c_2 = -4\pi^2,\ c_3 = -9\pi^2, \ldots,\ c_n = -n^2\pi^2,$$

Plugging this back to the expression of $G(x)$ for the case with $c < 0$, we obtain the **eigenfunctions**

$$G_0(x) = 0 \text{ (trivial)},\ G_1(x) = \sin(\pi x),\ G_2(x) = \sin(2\pi x), \ldots\ G_n(x) = \sin(n\pi x), \ldots$$

Lastly, for the case with $c = 0$, the ODE is reduced to $G''(x) = 0$, whose general solution is $G(x) = A x + B$. From b.c. (I), we have $B = 0$; From b.c (II), we have $A = 0$, so the solution is trivial.

Beware that for some eigenvalue problems the zero eigenvalue, c = 0, can correspond to a non-trivial solution. Don't dismiss the case outright!

Exercises: Solve the following eigenvalue problems

(i) $G''(x) = c\, G(x)$, $G(3) = 0$, $G(5) = 0$

(ii) $G''(x) = c\, G(x)$, $G(0) = 0$, $G'(1) = 0$ (G' is dG/dx)

(iii) $G''(x) = c\, G(x)$, $G'(0) = 0$, $G(1) = 0$

(iv) $G''(x) = c\, G(x)$, $G(0) = G(1)$, $G'(0) = G'(1)$ (periodic boundary condition)