Boundary conditions and eigenvalue problem

Example 1: Eigenvalue problem. Consider the equation, $d^2u/dx^2 = \lambda u$, where λ is yet undetermined. In this case, three different types of b.c.'s are generally valid: (i) $u(a) = A$, $u(b) = B$, (ii) $u(a) = A$, $u'(b) = B$, and (iii) $u'(a) = A$, $u'(b) = B$ They lead to different eigenvalues that would ensure a non-trivial solution for u.

Example 1A: Solve the eigenvalue problem, $d^2u/dx^2 = \lambda u$, $u(0) = 0$, $u(1) = 0$.

(i) If $\lambda > 0$, write $\lambda = k^2$, $k > 0$; the general solution of the ODE is

$$u(x) = C \sinh(kx) + D \cosh(kx) \quad [\text{Note: } \sinh(x) \equiv (e^x - e^{-x})/2 \text{ , } \cosh(x) \equiv (e^x + e^{-x})/2]$$

It can be shown that, to satisfy the b.c.'s, we can only have $D = 0$ and $C = 0$ (or $k = 0$) $\Rightarrow$ trivial solution $u(x) = 0$

(ii) If $\lambda < 0$, write $\lambda = -k^2$, $k > 0$; the general solution of the ODE is

$$u(x) = C \sin(kx) + D \cos(kx) \text{ .}$$

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(continued)

From the 1st b.c., $u(0) = 0 \Rightarrow C\sin(0) + D\cos(0) = 0 \Rightarrow D = 0$

From the 2nd b.c., $u(1) = 0 \Rightarrow C\sin(\mathrm{k}) + D\cos(\mathrm{k}) = 0 \Rightarrow C\sin(\mathrm{k}) = 0$

This means $C = 0$ or $\sin(\mathrm{k}) = 0$, the former leads to a trivial solution. Thus, to obtain non-trivial solution(s) we must have

$\sin(\mathrm{k}) = 0 \;\Rightarrow\; \mathrm{k} = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$ are the only allowed values of k that lead to non-trivial solutions. (k = 0 is excluded since it leads to trivial solution.)

Recall that $\lambda = -\mathrm{k}^2$, we obtain the *eigenvalues*,

$\lambda = \lambda_n = -(n\pi)^2$, $n = 1, 2, 3, \ldots$, (The λ_n with $n = -1, -2, \ldots$ are redundant and can be ignored)

and their corresponding *eigenfunctions*,

$u_n(x) = \sin(\mathrm{k}_n x) = \sin(n\pi x)$, $n = 1, 2, 3, \ldots$

Any pair of ($\lambda = \lambda_n$, $u = u_n(x)$) are a solution of the ODE+b.c.'s.

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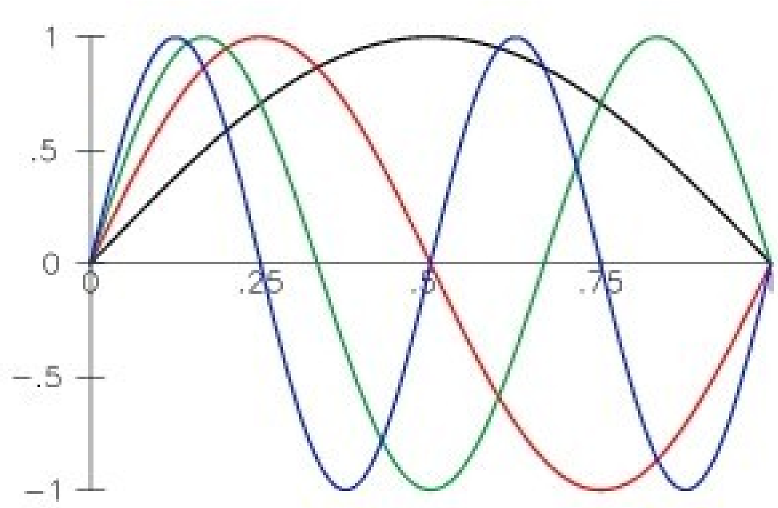
(continued)

The linear superposition of any combination of the eigenfunctions is also the solution of the ODE+b.c.'s. ⇒ The most general form of the solution is

$$u(x) = a_1\, u_1(x) + a_2\, u_2(x) + a_3\, u_3(x) + \ldots$$
$$= \sum_{n=1}^{\infty} a_n u_n(x) \quad .$$

The following figure shows the first 4 eigenfunctions (they are just sinusoidal functions)



Exercise: Solve the eigenvalue problem with the same ODE as Example 1A but with the boundary conditions $u'(0) = 0$, $u'(1) = 0$.