Boundary conditions for heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{(No heat source)}$$

<u>Separation of variables</u>

(Minimum detail for current discussion; to be revisited in later lectures)

Step 1: Assume that the solution can be written as $u(x,t) = \mathrm{G}(x)\mathrm{H}(t)$; Plugging it into the PDE leads to $G\frac{dH}{dt} = H\frac{d^2G}{dx^2}$

$$\Rightarrow \quad \frac{1}{H}\frac{dH}{dt} = \frac{1}{G}\frac{d^2G}{dx^2}$$

Step 2: Since the r.h.s. of the above equation depends only on x and l.h.s. depends only on t, the only possibility for the equation to hold is that r.h.s. = l.h.s. = a *common* constant; $\frac{1}{H}\frac{dH}{dt} = \frac{1}{G}\frac{d^2G}{dx^2} = C$. With this, the original PDE is transformed into a pair of ODEs

$$\frac{dH}{dt} = cH \qquad (1)$$

$$\frac{d^2G}{dx^2} = cG \qquad (2)$$

For Eq. (1), a valid b. c. is

(i) $H(t_0) = h$

For Eq. (2), the following are some meaningful b. c.'s ,

(ii) $G(x_1) = A$, $G(x_2) = B$, (iii) $G(x_1) = A$, $G'(x_2) = B$, (iii) $G'(x_1) = A$, $G'(x_2) = B$

Eq. (2) combined with one of (i)-(iii) form an eigenvalue problem.

Without loss of generality*, we may set $t_0 = 0$, $x_1 = 0$, and $x_2 = 1$. Since $u(x,t) = G(x)H(t)$, condition (i) can be translated to

(I) $u(x, 0) = F(x)$,

for the PDE, where $F(x)$ is a given function, and (ii) can be translated to

(II) $u(0, t) = P(t)$, (III) $u(1, t) = Q(t)$.

(continue to next slide)

*The solution of the equation defined on $x \in [a, b]$ can be readily obtained by a simple re-scaling of the solution of the equation defined on $x \in [0,1]$. Likewise, the change of variable, $t \rightarrow t + a$, is trivial.

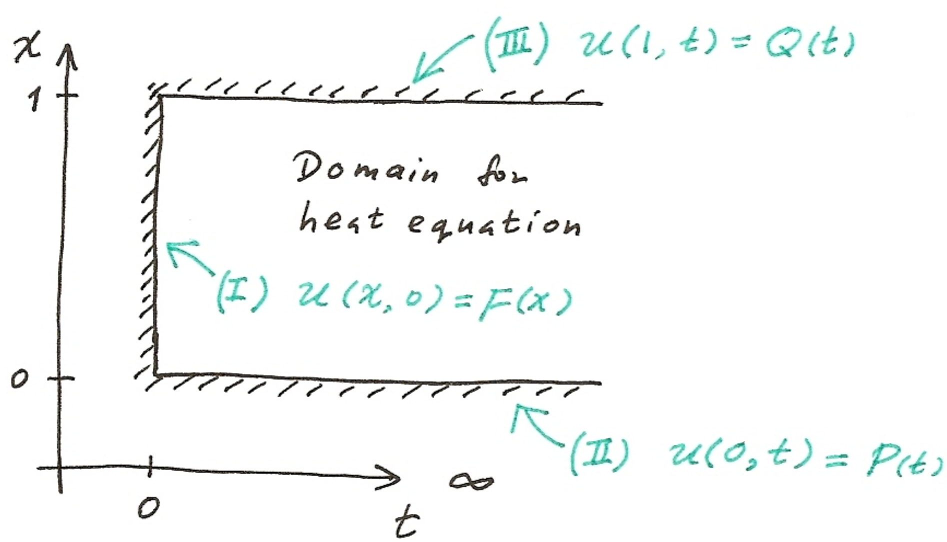
(continued)
Similarly, condition (ii) can be translated to

$$\text{(II')}\ u(0, t) = P(t)\ , \quad \text{(III')}\ \ u_x(1, t) = Q(t)\ , \text{ where } u_x \equiv \partial u/\partial x\ ,$$

and condition (iii) can be translated to

$$\text{(II'')}\ u_x(0, t) = P(t)\ , \quad \text{(III'')}\ \ u_x(1, t) = Q(t)\ .$$

The boundary conditions, {(I), (II), (III)} (see illustraion below), {(I), (II'), (III')}, or {(I), (II''), (III'')}, are imposed to the three "walls" of the semi-open domain in the t-x plane bounded by $t = 0$, $x = 0$, and $x = 1$. The solution satisfies the PDE within the domain, $x \in [0,1]$, $t \in [0, \infty)$, and satisfies the b.c.'s at the "walls".



In physical terms ...

- The $F(x)$ in condition (I) is the "initial state" of $u(x,t)$ at $t = 0$. It describes the distribution of temperature, u, along the metal rod in the heat transfer problem. This distribution then marches forward in time, with its evolution governed by the PDE.

- The case with $P(t)$ = constant in condition (II) (or $Q(t)$ = constant in (III)) is when the temperature at one (or both) ends of the metal rod are kept fixed. (This can be achieved by attaching the end of the rod to a thermal bath with a constant temperature.)

- Since heat flux, ϕ, is proportional to $-\partial u/\partial x$, condition (III') (or (II") and (III")) is equivalent to imposing the value of heat flux at one end of the rod. If $Q(t)$ = constant in (III'), it means the heat flux into (or out of) the metal rod through one end of the rod is held constant.