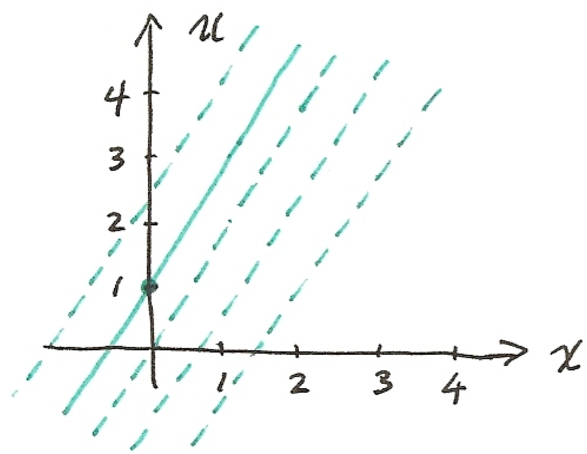
Boundary conditions - general remarks

ODE/PDE + boundary condition(s) $\Rightarrow$ complete solution
(ODE/PDE alone without b.c. $\Rightarrow$ General solution, not always useful)

Example 1: The ODE, $du/dx = 2$, has the general solution $u(x) = 2x + C$, where C is an arbitrary constant. This describes a family of lines in x-u plane with a slope of 2 (see figure below). A boundary condition, $u(0) = 1$, will add the constraint that the line must pass the point $(x, u) = (0, 1)$ $\Rightarrow C = 1$ $\Rightarrow$ unique solution $u(x) = 2x+1$ (solid line in the figure)



For more complicated problems, picking the right curve/surface that fits the b.c. is often more difficult than determining the general solution. **Boundary condition is important. An ODE or PDE combined with a wrong type of boundary condition(s) may lead to no solution at all.**

Examples of "healthy" and "unhealthy" boundary conditions

First order ODE

For the ODE in Example 1,

(i) Imposing two b.c.'s at two different x, e.g., $u(0) = 1$ and $u(1) = 2$, would lead to contradiction $\Rightarrow$ No solution. (Special cases such as $u(0) = 1$, $u(1) = 3$, would lead to a solution, but in this case the second b.c. is redundant.)

(ii) Imposing a single b.c. for u' (the first derivative) instead of u, e.g., $u'(0) = 3$, will also lead to contradiction $\Rightarrow$ No solution.

A healthy b.c. for this ODE must be of the form, $u(a) = A$., i.e., an "initial condition" for u given at a single point of x.

Second order ODE

Example 2: The ODE, $d^2u/dx^2 = 2$, has the general solution, $u(x) = x^2 + C\,x + D$, where C and D are arbitrary constants. Consider the following types of b.c.'s:

(i) $u(a) = A$, $u'(a) = B$. For example, $u(0) = 1$, $u'(0) = 1 \Rightarrow D = 1$, $C = 1 \Rightarrow$ unique solution $u(x) = x^2 + x + 1$.

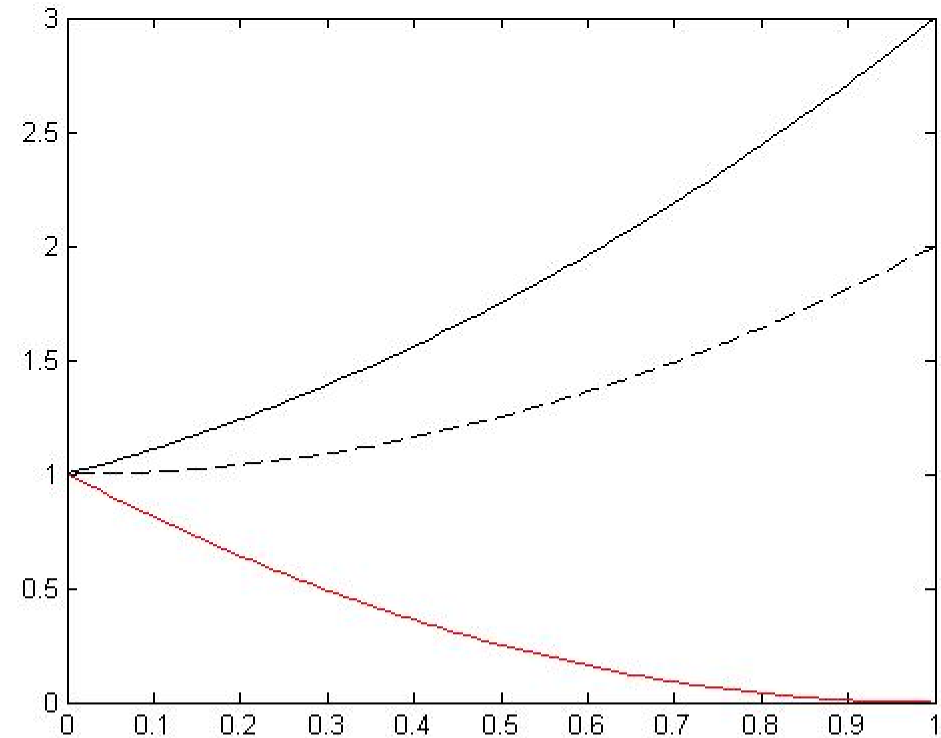
(ii) $u(a) = A$, $u(b) = B$, $a \neq b$. For example, $u(0) = 1$, $u(1) = 0 \Rightarrow D = 1$, $C = -2 \Rightarrow$ unique solution $u(x) = x^2 - 2x + 1$.

(iii) $u(a) = A$, $u'(b) = B$, $a \neq b$. For example, $u(0) = 1$, $u'(1) = 2 \Rightarrow D = 1$, $C = 0 \Rightarrow$ unique solution $u(x) = x^2 + 1$.

(iv) $u'(a) = A$, $u'(b) = B$. For example, $u'(0) = 1$, $u'(1) = 2 \Rightarrow$ "$1 = 0$", contradiction; solution does not exist. (Special cases such as $u'(0) = 1$, $u'(1) = 3$ would avoid contradiction. Yet, they do not lead to a useful solution since D remains undetermined.)

Types (i)-(iii) are the healthy b. c.'s for this ODE. (The conclusion is specific to this ODE. For a different ODE the situation may be different.)

The figure in next page shows the solutions from the examples in (i)-(iii). Note that all three curves satisfy the same ODE but different b. c.'s.



(i) black solid (ii) red (iii) black dashed