

Examples of numerical solutions for boundary value problems (for ODE)

Example 1

$$d^2u/dx^2 = 1 \quad , \quad u(0) = 0, u(1) = 0$$

Analytic solution is $u(x) = x^2/2 - x/2$. To obtain a numerical solution, the equation is first discretized on the grid, $[x_0, x_1, x_2, \dots, x_N]$, where $x_0 = 0$, $x_N = 1$, and $x_i = i \Delta x$. We will use $\Delta x = 0.2$ ($N = 5$) through the discussion. The second derivative can be approximated by the *three-point central difference* scheme (this is but one of many possible choices), which leads to

$$(u_{i-1} - 2u_i + u_{i+1})/(\Delta x)^2 = 1 \quad \text{for } i = 1, 2, 3, 4 \quad (1)$$

Note that these equations are only for the interior points, $x = x_1, x_2, x_3$, and x_4 (i.e., $x = 0.2, 0.4, 0.6$, and 0.8). The values of u at the boundary points, x_0 and x_5 , are given by the boundary conditions, $u_0 = 0$ and $u_5 = 0$. System (1) admits a unique solution since it has 4 unknowns (u_1, u_2, u_3, u_4) and 4 equations. Without the boundary conditions, one would have 6 unknowns but only 4 equations. The 4 equations in (1) are

$$\begin{aligned} u_0 - 2u_1 + u_2 &= (\Delta x)^2 \\ u_1 - 2u_2 + u_3 &= (\Delta x)^2 \\ u_2 - 2u_3 + u_4 &= (\Delta x)^2 \\ u_3 - 2u_4 + u_5 &= (\Delta x)^2 \quad , \end{aligned} \quad (2)$$

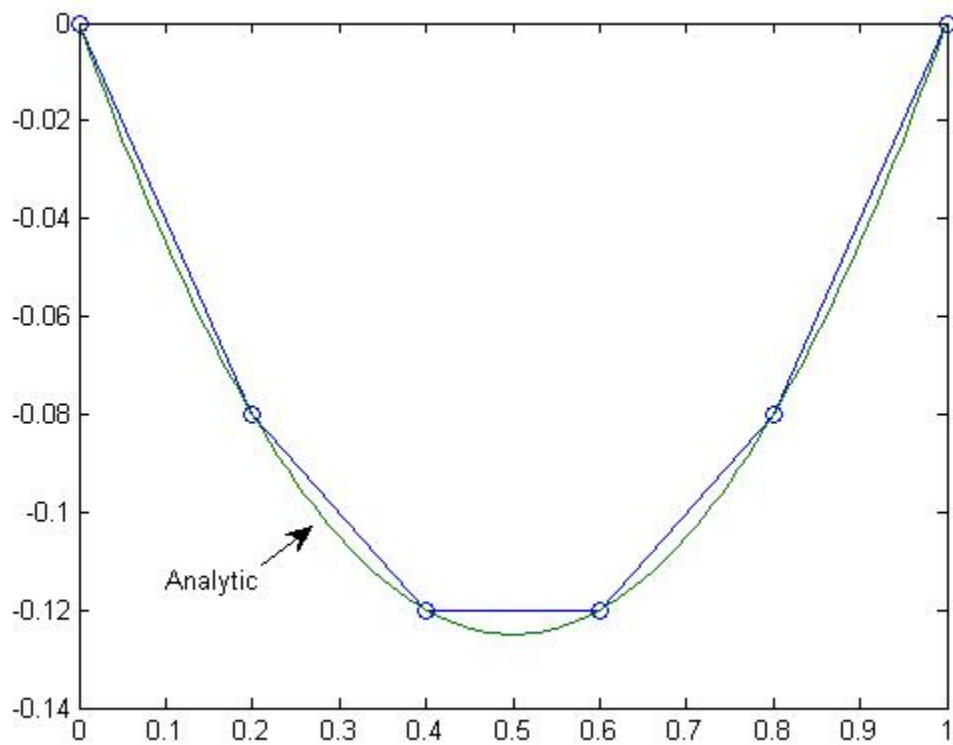
where $(\Delta x)^2 = 0.04$. The two red elements are associated with the boundary conditions and, in this case, can be readily eliminated. This leads to

(Ex. 1 continued)

$$\begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 0.04 \\ 0.04 \\ 0.04 \\ 0.04 \end{pmatrix} \quad (3)$$

Solving (3) for (u_1, u_2, u_3, u_4) , one obtains the complete solution, $(u_0, u_1, u_2, u_3, u_4, u_5) = (0, -0.08, -0.12, -0.12, -0.08, 0)$. This is plotted in the figure in next page (numerical solution in circles, analytic solution in solid curve). As it turns out, in this case the numerical solution is exact (no error) at all x_i . This is possible because the analytic solution itself is only a 2nd order polynomial.

The matrix in (3) is an example of a *tridiagonal matrix*. While one can always solve the system by Gauss elimination and other standard procedures, the fact that most of the off-diagonal elements are zero makes the solution particularly easy to obtain. We should next demonstrate a useful procedure to solve a tridiagonal system.



Result for Ex. 1 (blue is numerical solution, green is analytic solution)

Trick: Solving a system of linear equations with a tridiagonal matrix. To illustrate the procedure, we will use the matrix from previous example but a more general r.h.s. vector:

$$\begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} .$$

We will work from top row to the bottom. From the top equation,

$$-2 u_1 + u_2 = b_1 \implies u_2 = b_1 + 2 u_1 \quad (\text{i})$$

Using the 2nd equation and (i),

$$u_3 = b_2 - u_1 + 2 u_2 = b_2 + 2 b_1 + 3 u_1 \quad (\text{ii})$$

(The key is that we are writing all the u_i as a function of u_1 .) From the 3rd equation plus (i) and (ii),

$$u_4 = b_3 - u_2 + 2 u_3 = b_3 + 2 b_2 + 3 b_1 + 4 u_1 \quad (\text{iii})$$

Finally, at the bottom, we combine $u_3 - 2 u_4 = b_4$ with (ii) and (iii) to solve u_1 as

$$u_1 = (-4 b_1 - 3 b_2 - 2 b_3 - b_4) / 5$$

Once u_1 is solved, u_2 , u_3 , and u_4 can be determined by back substituting u_1 into (i), (ii), and (iii).

(Exercise: Try to use this procedure to solve Eq. (3) in Example 1.)

Example 2

$$d^2u/dx^2 = 1 \quad , \quad u(0) = 1, u(1) = 2$$

This is a slight modification of Ex. 1. Let's use $\Delta x = 0.2$ as before. Following Ex. 1, here one simply substitutes u_0 and u_5 by their non-zero values, 1 and 2 respectively, at the boundaries,

$$\begin{aligned} 1 - 2u_1 + u_2 &= (\Delta x)^2 \\ u_1 - 2u_2 + u_3 &= (\Delta x)^2 \\ u_2 - 2u_3 + u_4 &= (\Delta x)^2 \\ u_3 - 2u_4 + 2 &= (\Delta x)^2 \quad , \end{aligned}$$

which leads to

$$\begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} -0.96 \\ 0.04 \\ 0.04 \\ -1.96 \end{pmatrix} .$$

The rest of the procedure is the same as Ex. 1.

Example 3

$$d^2u/dx^2 = \sin(x) \quad u(0) = 0, u(\pi) = 0$$

The analytic solution is $u(x) = -\sin(x)$. For convenience of discussion, let's discretize the system with $\Delta x = 0.2\pi$. (Thus, $N = 5$ as before.) The procedure of discretization for this problem is similar to Ex. 1, except that the elements in the vector in the r.h.s. are replaced with non-constant values of $\sin(x_1)$, $\sin(x_2)$, etc. We now have

$$\begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = (0.2\pi)^2 \begin{pmatrix} \sin(0.2\pi) \\ \sin(0.4\pi) \\ \sin(0.6\pi) \\ \sin(0.8\pi) \end{pmatrix},$$

which can be readily solved to obtain the final numerical solution.

Example 4

$$d^2u/dx^2 = u, \quad u(0) = 0, \quad u(1) = 1 - \exp(-2)$$

The analytic solution is $u(x) = \exp(x-1) - \exp(-x-1)$. Using $\Delta x = 0.2$, the discretized system is

$$(u_{i-1} - 2u_i + u_{i+1})/(\Delta x)^2 - u_i = 0 \quad \text{for } i = 1, 2, 3, 4 \quad (4),$$

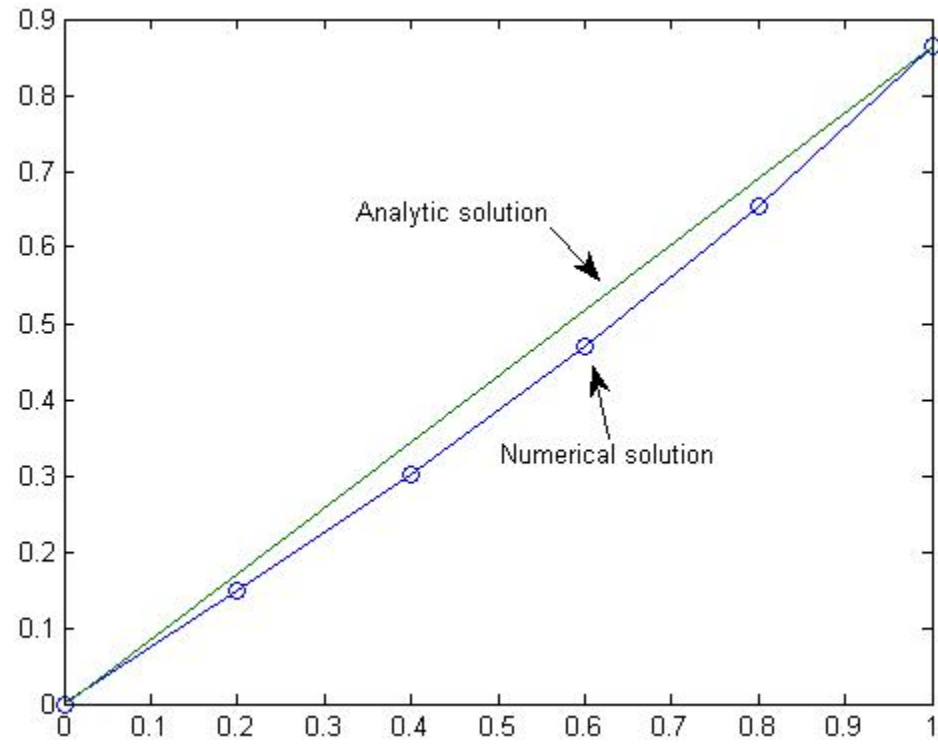
along with the boundary conditions $u_0 = 0$ and $u_5 = 1 - \exp(-2)$. Equation (4) can be re-organized into

$$P u_{i-1} + Q u_i + R u_{i+1} = 0 \quad \text{for } i = 1, 2, 3, 4, \quad \text{where } P = 1, \quad Q = -2 - (\Delta x)^2, \quad R = 1.$$

For $\Delta x = 0.2$, $Q = -2.04$. This leads to

$$\begin{pmatrix} -2.04 & 1 & 0 & 0 \\ 1 & -2.04 & 1 & 0 \\ 0 & 1 & -2.04 & 1 \\ 0 & 0 & 1 & -2.04 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ e^{-2}-1 \end{pmatrix} \quad (5)$$

A comparison of the numerical solution from (5) and the analytic solution is shown in the plot in next page.



Result of Example 4

What if a boundary condition involves the first derivative of u?

Example 5

$$d^2u/dx^2 = 1, \quad u'(0) = 0.5, \quad u(1) = 2$$

The analytic solution is $u(x) = x^2/2 + x/2 + 1$. Choosing $\Delta x = 0.2$ and following the same first step for Example 1, we obtain

$$\begin{aligned} u_0 - 2u_1 + u_2 &= (\Delta x)^2 \\ u_1 - 2u_2 + u_3 &= (\Delta x)^2 \\ u_2 - 2u_3 + u_4 &= (\Delta x)^2 \\ u_3 - 2u_4 + u_5 &= (\Delta x)^2, \end{aligned}$$

The second boundary condition is just $u_5 = 2$. To incorporate the first boundary condition, at $x = 0$ we approximate the first derivative by $u'(x_i) \approx (u_{i+1} - u_i)/\Delta x$ (the *forward difference* scheme), which leads to

$$u_1 - u_0 = 0.5 \Delta x \implies u_0 = u_1 - 0.5 \Delta x.$$

Then, the discrete system becomes

$$\begin{aligned} u_1 - 0.5 \Delta x - 2u_1 + u_2 &= (\Delta x)^2 \\ u_1 - 2u_2 + u_3 &= (\Delta x)^2 \\ u_2 - 2u_3 + u_4 &= (\Delta x)^2 \\ u_3 - 2u_4 + 2 &= (\Delta x)^2. \end{aligned} \tag{6}$$

With a slight rearrangement and using $(\Delta x)^2 = 0.04$, Eq. (6) becomes

(Ex. 5 continued)

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 0.14 \\ 0.04 \\ 0.04 \\ -1.96 \end{pmatrix},$$

which can be readily solved.

(Exercise: Finish the solution of the above matrix equation and compare the numerical solution with analytic solution.)