Application of Fourier Transform to PDE (II)

- Fourier Transform (application to PDEs defined on an infinite domain)

The Fourier Transform pair are

F. T. : $U(\omega)=(1/2\pi)\int_{-\infty}^{\infty} u(x)\exp(-i\omega x)dx$, denoted as $U = \mathbf{F}[u]$

Inverse F.T. : $u(x)=\int_{-\infty}^{\infty} U(\omega)\exp(i\omega x)d\omega$, denoted as $u = \mathbf{F}^{-1}[U]$

Note that $u = \mathbf{F}^{-1}[\mathbf{F}[u]]$, i.e., the successive action of Fourier transform and inverse Fourier transform brings the function $u(x)$ back to itself.

Example 1. Solve the 1-D heat equation,

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2} , \qquad (1)$$

for $u(x,t)$ defined on $x \in (-\infty, \infty)$, $t \in [0, \infty)$, with the boundary condition

(I) u(x,0) = P(x) (P(x) is the initial distribution of temperature) .

Note that (I) is the only b.c. we need. (Also, u and all of its derivatives in x vanish as x $\to \infty$.)

Step 1: Apply F. T. to the PDE, Eq. (1). Note that if $A(x) = B(x)$, then $\mathbf{F}[A] = \mathbf{F}[B]$. Thus, from Eq. (1),

$$\mathbf{F}\left[\frac{\partial u}{\partial t}\right] = \mathbf{F}\left[\frac{\partial^2 u}{\partial x^2}\right]. \qquad (2)$$

The l.h.s. of Eq. (2) is simply $\mathbf{F}\left[\frac{\partial u}{\partial t}\right] = \frac{\partial U}{\partial t}$, $U \equiv U(\omega,t)$. The r.h.s. of Eq. (2) can be rewritten as

$$\begin{aligned}
\mathbf{F}\left[\frac{\partial^2 u}{\partial x^2}\right] &= (1/2\pi)\int_{-\infty}^{\infty}\frac{\partial^2 u}{\partial x^2}\exp(-i\omega x)\,dx \\
&= (1/2\pi)\int_{-\infty}^{\infty}\exp(-i\omega x)\,d\left(\frac{\partial u}{\partial x}\right) \\
&= (1/2\pi)\left[\!\left[\left[\exp(-i\omega x)\frac{\partial u}{\partial x}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{\partial u}{\partial x}\,d(\exp(-i\omega x))\right]\!\right] \quad \text{(1st term in [ ] vanishes)} \\
&= (i\omega/2\pi)\int_{-\infty}^{\infty}\exp(-i\omega x)\left(\frac{\partial u}{\partial x}\right)dx \\
&= (i\omega/2\pi)\int_{-\infty}^{\infty}\exp(-i\omega x)\,du \\
&= (i\omega/2\pi)\left[\!\left[\left[\exp(-i\omega x)\,u\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}u\,d(\exp(-i\omega x))\right]\!\right] \quad \text{(1st term in [ ] vanishes)} \\
&= -(\omega^2/2\pi)\int_{-\infty}^{\infty}u\exp(-i\omega x)\,dx \\
&= -\omega^2\,U(\omega,t) \quad .
\end{aligned}$$

Thus, we obtain the equation for $U(\omega,t)$ as

$$\frac{\partial U(\omega,t)}{\partial t} = -\omega^2 U(\omega,t) \qquad (3)$$

Equation (3) has a simple solution,

$$U(\omega,t) = U(\omega,0)\exp(-\omega^2 t) \quad . \qquad (4)$$

[**Exercise**: Find the Fourier transform of the equation, $\partial u/\partial t = A\, \partial^4 u/\partial x^4$, and discuss the qualitative behavior of the solution for positive and negative A.]

Step 2b: Complete the solution of $U(\omega,t)$ with $U(\omega,0)$ determined from the initial state, u(x,0).
Note that $U(\omega,0)$ is the Fourier transform of the initial state in physical space, u(x,0),

$$U(\omega,0) = (1/2\pi)\int_{-\infty}^{\infty} u(x,0)\exp(-i\omega x)\,dx$$

$$= (1/2\pi)\int_{-\infty}^{\infty} P(x)\exp(-i\omega x)\,dx \quad . \qquad (5)$$

Step 3: Inverse F. T. of $U(\omega, t)$ gives the complete solution, u(x,t);

$$u(x,t) = \mathbf{F}^{-1}[U(\omega,t)] = \int_{-\infty}^{\infty} U(\omega,t)\exp(i\omega x)\,d\omega \quad . \qquad (6)$$

See relevant remarks below Eq. (12) in slides# 14.

Example 1A. In Example 1, find the solution if the initial temperature distribution (in b.c. (I)) is given by

$$P(x) = 1\,, \quad -1 \le x \le 1 \qquad (7)$$
$$= 0, \quad \text{otherwise.}$$

First, from Eq. (5) we have

$$U(\omega,0) = (1/2\pi)\int_{-\infty}^{\infty} P(x)\exp(-i\omega x)\,dx$$
$$= \sin(\omega)/(\omega\pi)\ .$$

The, from Eq. (4) we obtain

$$U(\omega, t) = [\sin(\omega)/(\omega\pi)]\ \exp(-\omega^2 t). \qquad (8)$$

From Eq. (6), the complete solution is

$$u(x,t) = \int_{-\infty}^{\infty} (\sin(\omega)/\omega\pi)\exp(-\omega^2 t)\exp(i\omega x)\,d\omega\quad . \qquad (9)$$

We may need numerical integration to evaluate u(x,t) using Eq. (9), but in its form (unless further simplication is possible) Eq. (9) can be regarded as the final solution.

[Exercise: Evaluate Eq. (9) numerically to obtain u(x,t) at selected t and plot them.]