Application of Fourier Transform to PDE (II)

● Fourier Transform (application to PDEs defined on an infinite domain)

The Fourier Transform pair are

F. T. : $U(\omega) = (1/2\pi)\int_{-\infty}^{\infty} u(x)\exp(-i\omega x)\,dx$, denoted as $U = \mathsf{F}[u]$

Inverse F.T. : $u(x) = \int_{-\infty}^{\infty} U(\omega)\exp(i\omega x)\,d\omega$, denoted as $u = \mathsf{F}^{-1}[U]$

Note that $u = \mathsf{F}^{-1}[\mathsf{F}[u]]$, i.e., the successive action of Fourier transform and inverse Fourier transform brings the function $u(x)$ back to itself.

Example 1. Solve the 1-D heat equation,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \ , \qquad (1)$$

for $u(x,t)$ defined on the infinite interval in x, $-\infty < x < \infty$, and (as usual) $0 \le t < \infty$, given the boundary condition

(I) u(x,0) = P(x) (P(x) is the initial distribution of "temperature") .
(Also, u and all of its derivatives vanish as x $\to \infty$. and x $\to -\infty$)

Step 1: Apply F. T. to the PDE. Note that A(x) = B(x) implies F [A] = F [B]. Thus, from Eq. (1),

$$\mathsf{F}\left[\frac{\partial u}{\partial t}\right] = \mathsf{F}\left[\frac{\partial^2 u}{\partial x^2}\right]. \tag{2}$$

The l.h.s. of Eq. (2) is F $\left[\frac{\partial u}{\partial t}\right] = \frac{\partial U}{\partial t}$,$U \equiv U(\omega,t)$. The r.h.s. of Eq. (2) can be rewritten as

$$\begin{aligned}
\mathsf{F}\left[\frac{\partial^2 u}{\partial x^2}\right] &= (1/2\pi)\int_{-\infty}^{\infty}\frac{\partial^2 u}{\partial x^2}\exp(-i\omega x)\,dx \\
&= (1/2\pi)\int_{-\infty}^{\infty}\exp(-i\omega x)\,d\left(\frac{\partial u}{\partial x}\right) \\
&= (1/2\pi)\left[\!\left[\left[\exp(-i\omega x)\frac{\partial u}{\partial x}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{\partial u}{\partial x}\,d(\exp(-i\omega x))\right]\!\right] \quad \text{(1st term in [ ] vanishes)} \\
&= (i\omega/2\pi)\int_{-\infty}^{\infty}\exp(-i\omega x)\left(\frac{\partial u}{\partial x}\right)dx \\
&= (i\omega/2\pi)\int_{-\infty}^{\infty}\exp(-i\omega x)\,du \\
&= (i\omega/2\pi)\left[\!\left[\left[\exp(-i\omega x)\,u\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}u\,d(\exp(-i\omega x))\right]\!\right] \quad \text{(1st term in [ ] vanishes)} \\
&= -(\omega^2/2\pi)\int_{-\infty}^{\infty}u\exp(-i\omega x)\,dx \\
&= -\omega^2\,U(\omega,t) \qquad (\text{or } -\omega^2\ \mathsf{F}[u]).
\end{aligned} \tag{3}$$

(An important simplification of this step is given at the end of this set of slides.)

Thus, we obtain the equation for $U(\omega,t)$ as

$$\frac{\partial U(\omega,t)}{\partial t} = -\omega^2\, U(\omega,t) \qquad (4)$$

Equation (4) has a simple solution,

$$U(\omega,t) = U(\omega,0)\exp(-\omega^2 t) \quad . \qquad (5)$$

Step 2: Complete the solution of $U(\omega,t)$ by determining $U(\omega,0)$ from the initial state, u(x,0). Note that $U(\omega,0)$ is the Fourier transform of the initial state in physical space, u(x,0),

$$\begin{aligned} U(\omega,0) &= (1/2\pi)\int_{-\infty}^{\infty} u(x,0)\exp(-i\omega x)\,dx \\ &= (1/2\pi)\int_{-\infty}^{\infty} P(x)\exp(-i\omega x)\,dx \quad . \end{aligned} \qquad (6)$$

Step 3: Inverse F. T. of $U(\omega, t)$ leads to the complete solution, u(x,t);

$$u(x,t) = \mathsf{F}^{-1}[U(\omega,t)] = \int_{-\infty}^{\infty} U(\omega,t)\exp(i\omega x)\,d\omega \quad . \qquad (7)$$

Numerical integration is usually needed to evaluate Eq. (7) in order to obtain the value of u(x,t) fro given (x,t). Note that while the integrand in Eq. (7) appears to be complex, the final outcome of the integration is always real.

Example 1A. Find the solution of the PDE in Example 1 when P(x) is given by

$$P(x) = 1\,, \quad -1 \le x \le 1 \qquad (8)$$
$$= 0, \quad \text{otherwise.}$$

First, from Eq. (6) we have

$$U(\omega,0) = (1/2\pi)\int_{-\infty}^{\infty} P(x)\exp(-i\omega x)\,dx$$
$$= \sin(\omega)/(\omega\pi)\,.$$

The, from Eq. (5) we obtain

$$U(\omega, t) = [\sin(\omega)/(\omega\pi)]\,\exp(-\omega^2 t). \qquad (9)$$

Lastly, from Eq. (7), the complete solution is

$$u(x,t) = \int_{-\infty}^{\infty} (\sin(\omega)/\omega\pi)\exp(-\omega^2 t)\exp(i\omega x)\,d\omega \quad . \qquad (10)$$

We may need numerical integration to evaluate u(x,t) using Eq. (10), but in its form Eq. (10) can be regarded as the final solution.

[**Exercise**: Evaluate Eq. (10) numerically to obtain u(x,t) at a selected t (e.g., t = 0.1) and plot the solution as a function of x.]

Important remark:

The derivation of Eq. (3) is cumbersome and will be even more so if we have a higher derivative (e.g., $\partial^4 u/\partial x^4$) in the PDE. Here, we will show that the derivation can be simplified. In fact, only the knowledge of the Fourier transform of the first derivative of u is needed in order to obtain the F.T. of all orders of derivative of u. Let's first obtain the Fourier transform of the first derivative:

$$
\begin{aligned}
\mathsf{F}\left[\frac{\partial u}{\partial x}\right] &= (1/2\pi)\int_{-\infty}^{\infty}\frac{\partial u}{\partial x}\exp(-i\omega x)\,dx \\
&= (1/2\pi)\int_{-\infty}^{\infty}\exp(-i\omega x)\,du \\
&= (1/2\pi)\left[\!\left[\,\left[\exp(-i\omega x)u\right]_{-\infty}^{\infty}-\int_{-\infty}^{\infty}u\,d(\exp(-i\omega x))\right]\!\right] \quad \text{(1st term in [ ] vanishes)} \\
&= (i\omega/2\pi)\int_{-\infty}^{\infty}u\,\exp(-i\omega x)\,dx \\
&= i\omega\ \mathsf{F}[u] \\
&= i\omega\, U(\omega,t) \quad .
\end{aligned}
$$

Suppose that we now want to derive $\mathsf{F}\left[\frac{\partial^2 u}{\partial x^2}\right]$. Define $A \equiv \frac{\partial u}{\partial x}$, we have

$$
\mathsf{F}\left[\frac{\partial^2 u}{\partial x^2}\right] = \mathsf{F}\left[\frac{\partial A}{\partial x}\right] = i\omega\ \mathsf{F}[A] = i\omega\ \mathsf{F}\left[\frac{\partial u}{\partial x}\right] = (i\omega)(i\omega)\ \mathsf{F}[u] = -\omega^2\ \mathsf{F}[u]\,,
$$

which is the final result in Eq. (3).

Extending this argument to higher derivatives, we immediately obtain

$$\mathsf{F}\left[\frac{\partial^p u}{\partial x^p}\right] = (i\omega)^p\,\mathsf{F}[u] = (i\omega)^p\,U(\omega,t)\,.$$

This is an extremely useful result to remember.

[**Exercise**: Find the Fourier transform of the equation, $\partial u/\partial t = K\,\partial^4 u/\partial x^4$, and discuss the qualitative behavior of the solution for positive and negative *K*.]