Application of Fourier Transform to PDE (I)

• Fourier Sine Transform (application to PDEs defined on a semi-infinite domain)

The Fourier Sine Transform pair are

F. T. : $U(\omega)=(2/\pi)\int_0^\infty u(x)\sin(\omega x)\,dx$, denoted as $U = \mathbf{S}[u]$

Inverse F.T. : $u(x)=\int_0^\infty U(\omega)\sin(\omega x)\,d\omega$, denoted as $u = \mathbf{S}^{-1}[U]$

Remarks:

(i) The F.T. and I.F.T. defined above are analogous to their counterparts for Fourier Sine series

F. T. : $A(n)=(2/L)\int_0^L u(x)\sin(n\pi x/L)\,dx$,

Inverse F. T. : $u(x)=\sum_{n=1}^\infty A(n)\sin(n\pi x/L)\Delta n$, where $\Delta n = 1$.

(ii) Note that $u = \mathbf{S}^{-1}[\mathbf{S}[u]]$, i.e., the successive action of Fourier transform and inverse Fourier transform brings the function $u(x)$ back to itself. As long as this is satisfied, the leading constants for the integrals in the F.T. and I.F.T are adjustable. For instance, the $(2/\pi)$ and 1 for F. T. and I. F. T. above can be replaced by $(2/\pi)^{1/2}$ for both.

Example 1. Solve the 1-D heat equation,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \ , \tag{1}$$

for $u(x,t)$ defined on $x \in [0, \infty)$, $t \in [0, \infty)$, with the boundary conditions

(I) u(0,t) = 0
(II) u(x,0) = P(x)   (P(x) is the initial distribution of temperature) .

Note that we do not need a specific b.c. for $u(\infty, t)$ but only require that, for a physically meaningful system, u and all of its derivatives in x vanish as $x \to \infty$. (See textbook for detail)

Step 1: Apply F. T. to the PDE, Eq. (1). Note that if $A(x) = B(x)$, then $\mathbf{S}\,[A] = \mathbf{S}\,[B]$. Thus, from Eq. (1),

$$\mathbf{S}\left[\ \frac{\partial u}{\partial t}\ \right] = \mathbf{S}\left[\ \frac{\partial^2 u}{\partial x^2}\ \right]. \tag{2}$$

The l.h.s. of Eq. (2) is

$$\mathbf{S}\left[\ \frac{\partial u}{\partial t}\ \right] = (2/\pi)\int_0^\infty \frac{\partial u}{\partial t}\sin(\omega x)\,dx \ = \ \frac{\partial}{\partial t}\left[(2/\pi)\int_0^\infty u\sin(\omega x)\,dx\right] \ = \ \frac{\partial U}{\partial t}\ . \tag{3}$$

Here, $U \equiv U(\omega,t)$.

The r.h.s. of Eq. (2) can be rewritten as

$$
\begin{aligned}
\mathbf{S}\left[\frac{\partial^2 u}{\partial x^2}\right] &= (2/\pi)\int_0^\infty \frac{\partial^2 u}{\partial x^2}\sin(\omega x)\,dx \\
&= (2/\pi)\int_0^\infty \sin(\omega x)\,d\left(\frac{\partial u}{\partial x}\right) \\
&= (2/\pi)\left[\!\left[\left[\sin(\omega x)\frac{\partial u}{\partial x}\right]_0^\infty - \int_0^\infty \frac{\partial u}{\partial x}\,d(\sin(\omega x))\right]\!\right] \quad \text{(1st term in [ ] vanishes)} \\
&= (-2\omega/\pi)\int_0^\infty \cos(\omega x)\left(\frac{\partial u}{\partial x}\right)dx \\
&= (-2\omega/\pi)\int_0^\infty \cos(\omega x)\,du \\
&= (-2\omega/\pi)\left[\!\left[\left[\cos(\omega x)\,u\right]_0^\infty - \int_0^\infty u\,d(\cos(\omega x))\right]\!\right] \quad \text{(1st term in [ ] = } -u(0,t)\text{)} \\
&= (2\omega/\pi)u(0,t) - (2\omega^2/\pi)\int_0^\infty u\sin(\omega x)\,dx \\
&= (2\omega/\pi)u(0,t) - \omega^2\,U(\omega,t) \quad . \qquad (4)
\end{aligned}
$$

Combining Eqs. (3) and (4), we obtain the F.T. of the original PDE in "spactral space" as

$$
\frac{\partial U(\omega,t)}{\partial t} = (2\omega/\pi)u(0,t) - \omega^2\,U(\omega,t) \qquad (5)
$$

This is Eq. (10.5.32) in textbook. Note, however, that if we wish to use Fourier Sine transform to solve a PDE, the only "good" b.c. at x = 0 is u(0,t) = 0 as given above, therefore the 1st term in the r.h.s. of Eq. (5) vanishes anyway. Textbook is wrong by insisting on using the b.c. of u(0,t) = g(t).

Step 2a: Solve $U(\omega,t)$ in spectral space. Given the above discussion, applying the b.c., u(0,t) = 0, to Eq. (5) we have

$$\frac{d\,U(\omega,t)}{d\,t} = -\omega^2\,U(\omega,t) \quad . \tag{6}$$

Note that, for a given ω, this is just an ODE. Equation (6) has a simple solution,

$$U(\omega,t) = U(\omega,0)\exp(-\omega^2 t) \quad . \tag{7}$$

Step 2b: Complete the solution of $U(\omega,t)$ with $U(\omega,0)$ determined from the initial state, u(x,0). The last piece of information is the initial state in the spectral space, $U(\omega,0)$, in Eq. (7). It is simply the Fourier Sine transform of the initial state in physical space, u(x,0),

$$\begin{aligned} U(\omega,0) &= (2/\pi)\int_0^\infty u(x,0)\sin(\omega x)\,dx \\ &= (2/\pi)\int_0^\infty P(x)\sin(\omega x)\,dx \quad . \end{aligned} \tag{8}$$

Step 3: Inverse F. T. of $U(\omega, t)$ gives the complete solution, u(x,t);

$$u(x,t) = \mathbf{S}^{-1}[U(\omega,t)] = \int_0^\infty U(\omega,t)\sin(\omega x)\,d\omega \quad . \tag{9}$$

Remarks:

(i) In Example 1, if the b.c. at $x = 0$, $u(0,t) = 0$, is replaced by $u(0,t) = A$ (A is a non-zero constant), then Fourier Sine transform will not work since it always leads to $u(x,t) = 0$ at $x = 0$. In this case, Fourier Cosine transform can be used. [**Exercise**: Try to derive the counterpart of Eq. (5) when Fourier Cosine transform is used.]

(ii) In Example 1, most of the time the integral in Eq. (9) cannot be further reduced to a closed-form analytic expression. We would then leave it as is and regard Eq. (9) as the final solution. (Just like the Fourier series solution to a PDE defined on a finite interval. Often, the infinite series itself is the final solution we can have.) Numerical integration will be needed to evaluate Eq. (9) if we wish to determine the value of $u(x,t)$ at give (x,t).

Example 1A. In Example 1, find the solution if the initial temperature distribution (in b.c. (II)) is given by

$$P(x) = 1\,, \quad 1 \le x \le 2 \qquad (10)$$
$$= 0, \quad \text{otherwise.}$$

First, from Eq. (8) we have

$$U(\omega,0) = (2/\pi)\int_0^\infty P(x)\sin(\omega x)\,dx$$
$$= (2/\omega\pi)[\cos(\omega) - \cos(2\omega)]\ .$$

Using it in Eq. (7) we obtain

$$U(\omega, t) = (2/\omega\pi)[\cos(\omega) - \cos(2\omega)]\ \exp(-\omega^2 t). \qquad (11)$$

The complete solution is

$$u(x,t) = \int_0^\infty (2/\omega\pi)(\cos(\omega) - \cos(2\omega))\exp(-\omega^2 t)\sin(\omega x)\,d\omega\ . \qquad (12)$$

The integral in Eq. (12) likely cannot be simplified further. It can be regarded as the final form of the solution. The integral can be evaluated numerically to obtain the value of u(x,t) at given (x,t). To do so, note that although the range of integration is from 0 to ∞, $U(\omega,t)$ generally decays with ω so one can "truncate" the integral at a certain finite (but large enough) value of ω. (This is analogous to truncating the Fourier series solution at a finite value of *n*.)