

Application of Fourier Transform to PDE (I)

- Fourier Sine Transform (application to PDEs defined on a semi-infinite domain)

The Fourier Sine Transform pair are

$$\text{F. T. : } U(\omega) = (2/\pi) \int_0^{\infty} u(x) \sin(\omega x) dx, \text{ denoted as } U = \mathbf{S}[u]$$

$$\text{Inverse F.T. : } u(x) = \int_0^{\infty} U(\omega) \sin(\omega x) d\omega, \text{ denoted as } u = \mathbf{S}^{-1}[U]$$

Remarks:

- (i) The F.T. and I.F.T. defined above are analogous to their counterparts for Fourier Sine series

$$\text{F. T. : } A(n) = (2/L) \int_0^L u(x) \sin(n\pi x/L) dx,$$

$$\text{Inverse F. T. : } u(x) = \sum_{n=1}^{\infty} A(n) \sin(n\pi x/L) \Delta n, \text{ where } \Delta n = 1.$$

- (ii) Note that $u = \mathbf{S}^{-1}[\mathbf{S}[u]]$, i.e., the successive action of Fourier transform and inverse Fourier transform brings the function $u(x)$ back to itself. As long as this is satisfied, the leading constants for the integrals in the F.T. and I.F.T are adjustable. For instance, the $(2/\pi)$ and 1 for F. T. and I. F. T. above can be replaced by $(2/\pi)^{1/2}$ for both.

Example 1. Solve the 1-D heat equation,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} , \quad (1)$$

for $u(x,t)$ defined on the semi-infinite interval in x , $0 \leq x < \infty$, and (as usual) semi-infinite interval in t , $0 \leq t < \infty$, with the boundary conditions

$$(I) u(0,t) = 0$$

$$(II) u(x,0) = P(x) , \quad P(x) \text{ satisfies } P(0) = 0.$$

Note that we do not need a specific b.c. for $u(\infty, t)$ but only require that, for a physically meaningful system, u and all of its derivatives in x vanish as $x \rightarrow \infty$. (See textbook for detail)

Step 1: Apply F. T. to the PDE. Note that $A(x) = B(x)$ implies $S [A] = S [B]$. So, from Eq.(1),

$$S \left[\frac{\partial u}{\partial t} \right] = S \left[\frac{\partial^2 u}{\partial x^2} \right]. \quad (2)$$

The l.h.s. of Eq. (2) is

$$S \left[\frac{\partial u}{\partial t} \right] = (2/\pi) \int_0^{\infty} \frac{\partial u}{\partial t} \sin(\omega x) dx = \frac{\partial}{\partial t} \left[(2/\pi) \int_0^{\infty} u \sin(\omega x) dx \right] = \frac{\partial U}{\partial t} . \quad (3)$$

Here, $U \equiv U(\omega,t)$.

The r.h.s. of Eq. (2) can be rewritten as

$$\begin{aligned}
 \mathcal{S} \left[\frac{\partial^2 u}{\partial x^2} \right] &= (2/\pi) \int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin(\omega x) dx \\
 &= (2/\pi) \int_0^\infty \sin(\omega x) d\left(\frac{\partial u}{\partial x}\right) \\
 &= (2/\pi) \left[\left[\sin(\omega x) \frac{\partial u}{\partial x} \right]_0^\infty - \int_0^\infty \frac{\partial u}{\partial x} d(\sin(\omega x)) \right] \quad (\text{1st term in [] vanishes}) \\
 &= (-2\omega/\pi) \int_0^\infty \cos(\omega x) \left(\frac{\partial u}{\partial x}\right) dx \\
 &= (-2\omega/\pi) \int_0^\infty \cos(\omega x) du \\
 &= (-2\omega/\pi) \left[\left[\cos(\omega x) u \right]_0^\infty - \int_0^\infty u d(\cos(\omega x)) \right] \quad (\text{1st term in []} = -u(0,t)) \\
 &= (2\omega/\pi)u(0,t) - (2\omega^2/\pi) \int_0^\infty u \sin(\omega x) dx \\
 &= (2\omega/\pi)u(0,t) - \omega^2 U(\omega, t) \quad . \quad (4)
 \end{aligned}$$

Combining Eqs. (3) and (4), we obtain the F.T. of the original PDE in "spectral space" as

$$\frac{\partial U(\omega, t)}{\partial t} = (2\omega/\pi)u(0,t) - \omega^2 U(\omega, t) \quad (5)$$

This is Eq. (10.5.32) in textbook. Note, however, that **if we wish to use Fourier Sine transform to solve a PDE, the only "good" b.c. at $x = 0$ is $u(0,t) = 0$** , therefore the 1st term in the r.h.s. of Eq. (5) vanishes anyway. Textbook is wrong by insisting on using the b.c. of $u(0,t) = g(t)$.

Step 2: Solve $U(\omega,t)$ in spectral space. From the above discussion, applying the b.c., $u(0,t) = 0$, to Eq. (5) we obtain

$$\frac{d U(\omega, t)}{d t} = -\omega^2 U(\omega, t) \quad . \quad (6)$$

For a given ω , this is just an ODE with the simple solution,

$$U(\omega, t) = U(\omega, 0) \exp(-\omega^2 t) \quad . \quad (7)$$

Step 2: Complete the solution of $U(\omega,t)$ by determining $U(\omega,0)$ from the initial state, $u(x,0)$. In Eq. (7), the last piece of missing information is the initial state in the spectral space, $U(\omega,0)$. It is simply the Fourier Sine transform of the initial state in physical space, $u(x,0)$,

$$\begin{aligned} U(\omega, 0) &= (2/\pi) \int_0^{\infty} u(x, 0) \sin(\omega x) dx \\ &= (2/\pi) \int_0^{\infty} P(x) \sin(\omega x) dx \quad . \end{aligned} \quad (8)$$

Step 3: Inverse F. T. of $U(\omega, t)$ gives the complete solution, $u(x,t)$;

$$u(x,t) = \mathbf{S}^{-1}[U(\omega,t)] = \int_0^{\infty} U(\omega,t) \sin(\omega x) d\omega \quad . \quad (9)$$

Remarks:

(i) In Example 1, if $u(0,t) \neq 0$ and $P(0) \neq 0$, it would be inappropriate to use Fourier Sine transform. Instead, Fourier Cosine transform should be used. [**Exercise:** In this case, try to work out the detail of Fourier Cosine transform for the counterparts of Eqs. (5) and (9).]

(ii) In Example 1, unless $P(x)$ is exceedingly simple, the integral in Eq. (9) usually cannot be further reduced to a closed-form analytic expression. Numerical integration will be needed to evaluate Eq. (9) if we wish to determine the value of $u(x,t)$ at give (x,t) . This would require "truncating" the integral at a finite ω . *Analogy to Fourier series:* Recall that when we solve a PDE defined on a finite interval by Fourier series expansion, the final solution is in the form of an infinite series. In that case, in order to evaluate $u(x,t)$, we would have to truncate the infinite series at a finite n .

(iii) The derivation of Eq. (4) is rather cumbersome. It will be even more so if we have a higher derivative (e.g., $\partial^4 u / \partial x^4$) in the PDE. If we instead use the *complex Fourier transform* to treat the PDE, it will simplify the derivation. See the accompanying set of slides (Part II of the discussion on Fourier transform) for detail.

Example 1A. Find the solution of the heat equation in Example 1 when $P(x)$ is given by

$$\begin{aligned} P(x) &= 1, \quad 1 \leq x \leq 2 \\ &= 0, \quad \text{otherwise.} \end{aligned} \tag{10}$$

First, from Eq. (8) we have

$$\begin{aligned} U(\omega, 0) &= (2/\pi) \int_0^{\infty} P(x) \sin(\omega x) dx \\ &= (2/\omega\pi) [\cos(\omega) - \cos(2\omega)]. \end{aligned}$$

Using it in Eq. (7) we obtain

$$U(\omega, t) = (2/\omega\pi) [\cos(\omega) - \cos(2\omega)] \exp(-\omega^2 t). \tag{11}$$

The complete solution is

$$u(x, t) = \int_0^{\infty} (2/\omega\pi) (\cos(\omega) - \cos(2\omega)) \exp(-\omega^2 t) \sin(\omega x) d\omega. \tag{12}$$

The integral in Eq. (12) likely cannot be simplified further. It can be regarded as the final form of the solution. The integral can be evaluated numerically to obtain the value of $u(x, t)$ at given (x, t) . To do so, note that although the range of integration is from 0 to ∞ , $U(\omega, t)$ generally decays with ω so one can "truncate" the integral at a certain finite (but large enough) value of ω .