General remarks

1. General strategy for solving a complicated mathematical equation:

Transform it to a set of simpler equations that we already know how to solve

PDE $\Rightarrow$ ODE $\Rightarrow$ Algebraic equation

Example 1: Solve the ODE, $\dfrac{d^2 u}{d x^2} - 3 \dfrac{d u}{d x} + 2 u = 0$

Assume that $u \propto \exp(\alpha x)$ (This is an insight that mathematicians gave us - let's take it)
the ODE is transformed into an algebraic equation,

$$\alpha^2 - 3 \alpha + 2 = 0$$

$\Rightarrow \alpha = 1, 2 \Rightarrow$ two solutions : $u = \exp(x)$, $u = \exp(2 x)$

The general solution is $u(x) = A \exp(x) + B \exp(2 x)$
(A and B are constants that can be determined by boundary conditions)

2. What's in a solution ?

In the preceding example, the solution, $u = A\ \exp(x) + B\ \exp(2\ x)$, is useful because we can readily **use it to evaluate *u* for a given *x***. After all, this is our purpose of solving the equation.

The solution, expressed in terms of $\exp(x)$, is considered satisfactory because the exponential function is well understood and easy to evaluate, e.g., by using the power series representation, $\exp(x) = 1 + x + x^2/2! + x^3/3! + x^4/4! + \ldots$ We do not have to worry about this detail as our calculator or mathematical software always has a built-in routine for it.

[For example, $\exp(0.2) = 1 + 0.2 + 0.04/2 + 0.008/6 + 0.0016/24 + \ldots \rightarrow 1.221402\ldots$]

Example 2. Speaking of exponential function ...

Note that exp(x) is itself the solution of the ODE, du/dx = u (This may serve as a definiton of the exponential function.). We should see how the power series expression of exp(x) is obtained from this ODE.

Step 1: A well-behaved* function $u(x)$ can be expressed in power series (where the coefficients a_0, a_1, a_2, ..., are yet unknown)

$$u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots \qquad (1)$$

Step 2: Term-by-term differentiation of Eq. (1) yields

$$u'(x) = a_1 + 2\, a_2 x + 3\, a_3 x^2 + 4\, a_4 x^3 + \ldots \qquad (2)$$

Step 3: Plugging Eqs. (1) and (2) into the original ODE, $u' = u$, we obtain

$$(a_1 - a_0) + (2\, a_2 - a_1)\, x + (3\, a_3 - a_2)\, x^2 + (4\, a_4 - a_3)\, x^3 + \ldots = 0 \qquad (3)$$

For Eq. (3) to hold for all x, we must have

(continued)

*See a typical textbook of calculus for detail.

(continued)

$$a_1 - a_0 = 0\,, \qquad \text{(i)}$$
$$2\,a_2 - a_1 = 0\,, \qquad \text{(ii)}$$
$$3\,a_3 - a_2 = 0\,, \qquad \text{(iii)}$$
$$4\,a_4 - a_3 = 0\,, \qquad \text{(iv)}$$
$$\dots \quad \dots$$

and so on.

From (i) $\Rightarrow a_1 = a_0$

From (ii) $\Rightarrow a_2 = a_1/2 = a_0/2 = a_0/2!$

From (iii) $\Rightarrow a_3 = a_2/3 = a_0/(3 \times 2) = a_0/3!$

From (iv) $\Rightarrow a_4 = a_3/4 = a_0/(4 \times 3 \times 2) = a_0/4!$

...

... $\Rightarrow a_N = a_0/N!$

Step 4: Inserting the above a_1, a_2, ..., to Eq. (1), one obtains the solution to the ODE,

$$u(x) = a_0\ (1 + x + x^2/2! + x^3/3! + x^4/4! + \dots)\,,$$

which is exactly the exponential function.

(Here, a_0 can be determined by the initial condition. For example, with $u(0) = 1$ we have $a_0 = 1$.)

Observations:

In Example 2, a set of simple algebraic functions, x , x^2 , x^3 , ..., are used as the building blocks for the solution of the ODE, $u' = u$. We call the solution the exponential function, $\exp(x)$.

The exponential function, in turn, is used in Example 1 as the building block for the solution of a more complicated ODE, $u'' - 3\,u' + 2 = 0$.

In Example 2, the power series representation is useful because we know the basic properties of the algebraic functions (x, x^2, x^3, ...) very well. Specifically, we know their first derivatives as $d(x^N)/dx = N\,x^{N-1}$. Using this knowledge and term-by-term differentiation, the original ODE is converted to a set of algebraic equations.

A useful method of solving an ODE/PDE of u is to express u in *Fourier series*. In this case, we use a set of sinusoidal functions (e.g., $\sin(x)$, $\sin(2x)$, $\sin(3x)$, ...) as the building blocks for the solution of the equation. Just like x^N in the power series, term-by-term differentiation of $\sin(n\,x)$ is straightforward. We will return to this theme later.

Exercise: Use the method of power series expansion in Example 2 to solve the equation in Example 1. Your power series solution should turn out to be identical to $u = A\exp(x) + B\exp(2x)$.