Equilibrium solution to heat equation; Laplace equation

1-D heat equation

Recall (from Slides #9) that the general behavior of the solution to heat equation (without an internal heat source) is that the temperature profile becomes smoother with time; The magnitude of temperature gradient and heat flux decreases with an increasing t. As $t \to \infty$, the process of heat transfer hits diminishing return - no more moving around of heat energy $\Rightarrow \partial u/\partial t \to 0 \Rightarrow$ The original heat equation is reduced to $d^2u/dx^2 = 0$ (plus b.c.'s). This is the equation that describes the "end state" of the heat transfer process, i.e., $u(x,t)$ as $t \to \infty$ where $u(x, t)$ is the solution to the full time-dependent heat equation.

- The equilibrium solution is easier to obtain since it is governed by an ODE instead of a PDE.

- The equilibrium solution provides quick insights into the behavior of the solution (of the full heat equation) at large time; useful for checking the integrity of the full solution. As t increases, the full solution $u(x, t)$ should become closer and closer to the equilibrium solution. (If not, the full solution may be wrong.)

Example 1:

For $u(x, t)$ defined on $x \in [0, 1]$ and $t \in [0, \infty)$, find the full solution to the heat equation, $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, with b.c.'s: (I) $u(0, t) = 0$, (II) $u(1, t) = 0$, (III) $u(x, 0) = \sin(\pi x)$. Then, find the equilibrium solution $u_{eq}(x)$ and verify that $u(x, t) \to u_{eq}(x)$ as $t \to \infty$.

Solution:

Based on previous discussions in Slides #8 (or pp. 38-48 in textbook), the solution $u(x,t)$ that satisfies the PDE plus b.c.'s (I), (II), and (III) is

$$u(x, t) = \sin(\pi x) \exp(-\pi^2 t) .$$

The equilibrium state satisfies the ODE and the b.c.'s

$$d^2u/dx^2 = 0 \quad , \quad \text{(i)} \; u(0) = 0, \; \text{(ii)} \; u(1) = 0 \; .$$

[Note : (i) and (ii) correspond to (I) and (II) for the original PDE; b.c. (III) is not needed. As $t \to \infty$ the initial state is "forgotten" and is irrelevant to the equilibrium solution. All initial states converge to the same equilibrium solution as $t \to \infty$.]

The general solution to the ODE is $u(x) = C x + D$, but from b.c. (i) and (ii) we must have $C = 0$ and $D = 0 \Rightarrow u_{eq}(x) = 0$. (Temperature becomes zero everywhere.) Lastly, the time dependent solution $u(x, t)$ does approach $u_{eq}(x)$ since $\exp(-\pi^2 t) \to 0$ as $t \to \infty$.

Example 2:

Find the equilibrium solution to the problem described by the heat equation, $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$,

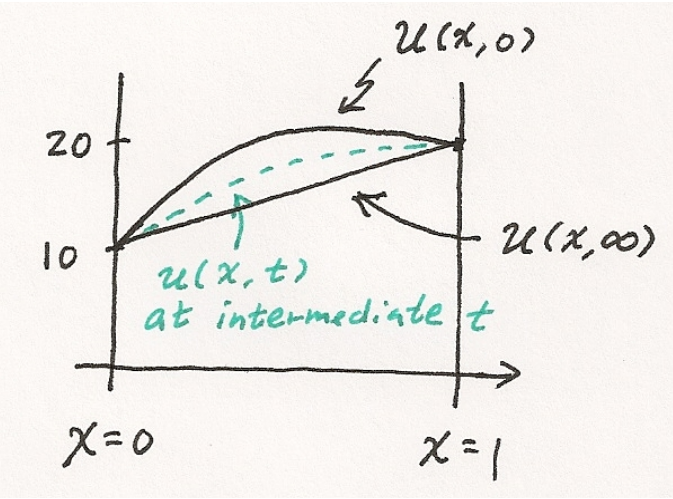
and b.c.'s: (I) $u(0, t) = 10$, (II) $u(1, t) = 20$, (III) $u(x, 0) = 10 + 10x + 5\sin(\pi x)$.

Solution:

The equilibrium state satisfies

$$d^2u/dx^2 = 0 \quad , \quad \text{(i)}\ u(0) = 10,\ \text{(ii)}\ u(1) = 20 .$$

The general solution to the ODE is $u(x) = C\,x + D$ but from the b.c.'s we have $C = 10$ and $D = 10 \Rightarrow u_{eq}(x) = 10 + 10\,x$. (Temperature distribution becomes a straight line.) The following is a sketch of the initial state and the equilibrium solution to the problem. Although we did not attempt to find the full time dependent solution, we can more or less infer what happens between $t = 0$ and $t \to \infty$.



Equilibrium solution to 2-D and 3-D heat equation; Laplace equation

The 2-D heat equation reads

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad .$$

With well-defined b.c.'s, $\partial u/\partial t \to 0$ as $t \to \infty$. The equilibrium state is governed by a time-independent equation, called Laplace equation in 2-D,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad .$$

Similarly, for the 3-D heat equation,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \quad ,$$

the corresponding equilibrium state satisfies the 3-D Laplace equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad .$$