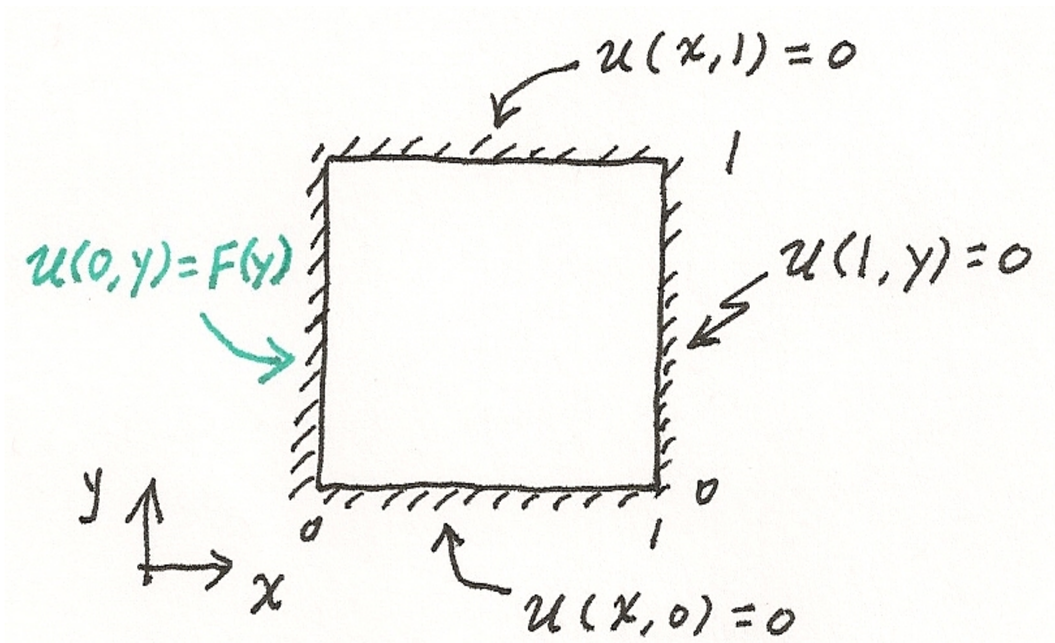
Example of an end-to-end solution to Laplace equation

Example 1: Solve Laplace equation, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, with the boundary conditions:

(I) $u(x, 0) = 0$ (II) $u(x,1) = 0$ (III) $u(0,y) = F(y)$ (IV) $u(1,y) = 0$.

See illustration below. This describes the equilibrium distribution of temperature in a slab of metal with the temperature at the top, bottom, and right sides fixed at u = 0. At the left side, the temperature is fixed to a certain non-zero temperature distribution, F(y), as a function of y. (This is similar to the problem discussed in **Sec. 2.5.1, pp. 71-75** in textbook, but note that we will have a more clear explanation of the point between Eq. (2.5.24) and Eq. (2.5.25) in p. 74.)



Step 1: Separation of variables, $u(x,y) = G(x)H(y)$, leads to

$$d^2G/dx^2 = -c\,G \quad , \quad \text{(iv)} \;\; G(1) = 0 \qquad (1)$$

$$d^2H/dy^2 = c\,H \quad , \quad \text{(i)} \;\; H(0) = 0 \;\; , \;\; \text{(ii)} \; H(1) = 0 \;\; , \qquad (2)$$

with c as a common constant for both equations. The b.c.'s (iv), (i), and (ii) correspond to (IV), (I), and (II) for the original PDE. The b.c. (III) is not readily separable and will be dealt with later.

Step 2: Solve eigenvalue problem in Eq.(2).

If $c > 0$, write $c = k^2$. The general solution is $H(y) = A \sinh(ky) + B \cosh(ky)$. It can be readily shown that the b.c.'s (i) and (ii) lead to $A = B = 0$ or $A = k = 0 \Rightarrow$ trivial solution.

If $c < 0$, write $c = -k^2$. The general solution is $H(y) = A \sin(ky) + B \cos(ky)$. Boundary condition (i) leads to $B = 0$, while b.c. (ii) leads to $\sin(k) = 0$,

$$\Rightarrow k_n = n\pi \;, n = 1, 2, 3, \ldots \qquad c_n = -(k_n)^2 \text{ are the eigenvalues.}$$

The corresponding eigenfunctions are

$$H_n(y) = \sin(k_n y) = \sin(n\pi y). \qquad (3)$$

Step 3: Set $c = c_n = -(k_n)^2$ to solve Eq. (1). Since $-c$ is positive, the solution must take the form of the combination of (sinh, cosh),

$$\begin{aligned} G_n(x) &= A \sinh(k_n x) + B \cosh(k_n x) \\ &= A \sinh(n\pi x) + B \cosh(n\pi x) \end{aligned} \qquad (4)$$

To satisfy b.c. (iv), we have

$$A \sinh(n\pi) + B \cosh(n\pi) = 0$$

$$\Rightarrow \; B/A = -\sinh(n\pi)/\cosh(n\pi).$$

So, we can rewrite Eq. (4) as

$$G_n(x) = A \sinh(n\pi x) - A[\sinh(n\pi)/\cosh(n\pi)] \cosh(n\pi x) \quad . \qquad (5)$$

Multiply a constant, $A^{-1} \cosh(n\pi)$, to Eq. (5) and it remains a solution to Eq. (1). This leads to

$$G_n(x) = \cosh(n\pi)\sinh(n\pi x) - \sinh(n\pi)\cosh(n\pi x) \quad . \qquad (6)$$

Using the identity,

$$\sinh(B-A) = \cosh(A)\sinh(B) - \sinh(A)\cosh(B),$$

Equation (6) can be rearranged into,

$$G_n(x) = \sinh(n\pi(1-x)) \ . \tag{7}$$

[This explains how the expression, $\sinh(L-x)$ is obtained in p. 74, between Eqs. (2.5.24) and (2.5.25), in the textbook.]

Step 3: Combine Eqs. (3) and (7) we have

$$u_n(x,y) = \sinh(n\pi(1-x))\sin(n\pi y) \ . \tag{8}$$

The most general form of the full solution is

$$u(x, y) = \sum_{n=1}^{\infty} a_n u_n(x, y) \ . \tag{9}$$

Step 4: We can now deal with the inseparable b.c. (III), $u(0,y) = F(y)$.

From Eq. (9) we have

$$u(0, y) = \sum_{n=1}^{\infty} a_n u_n(0, y) = \sum_{n=1}^{\infty} a_n \sinh(n\pi)\sin(n\pi y) \qquad (10)$$

Therefore, from b.c. (III),

$$F(y) = \sum_{n=1}^{\infty} a_n \sinh(n\pi)\sin(n\pi y) \ . \qquad (11)$$

Multiply both sides of Eq. (11) by sin(mπy), integrate both sides from 0 to 1 and incorporate the orthogonality relationship for sin(nπy), we obtain

$$a_m = \frac{2}{\sinh(m\pi)} \int_0^1 F(y)\sin(m\pi y)\,dy \ . \qquad (12)$$

Equations (9) and (12) form our complete solution.

Example 2: Solve the problem in Example 1 but with a specific b.c. :

(III) $u(0,y) = F(y)$, where

$$F(y) = y\,, \qquad 0 \le y \le 1/2$$
$$F(y) = 1 - y\,, \quad 1/2 \le y \le 1$$

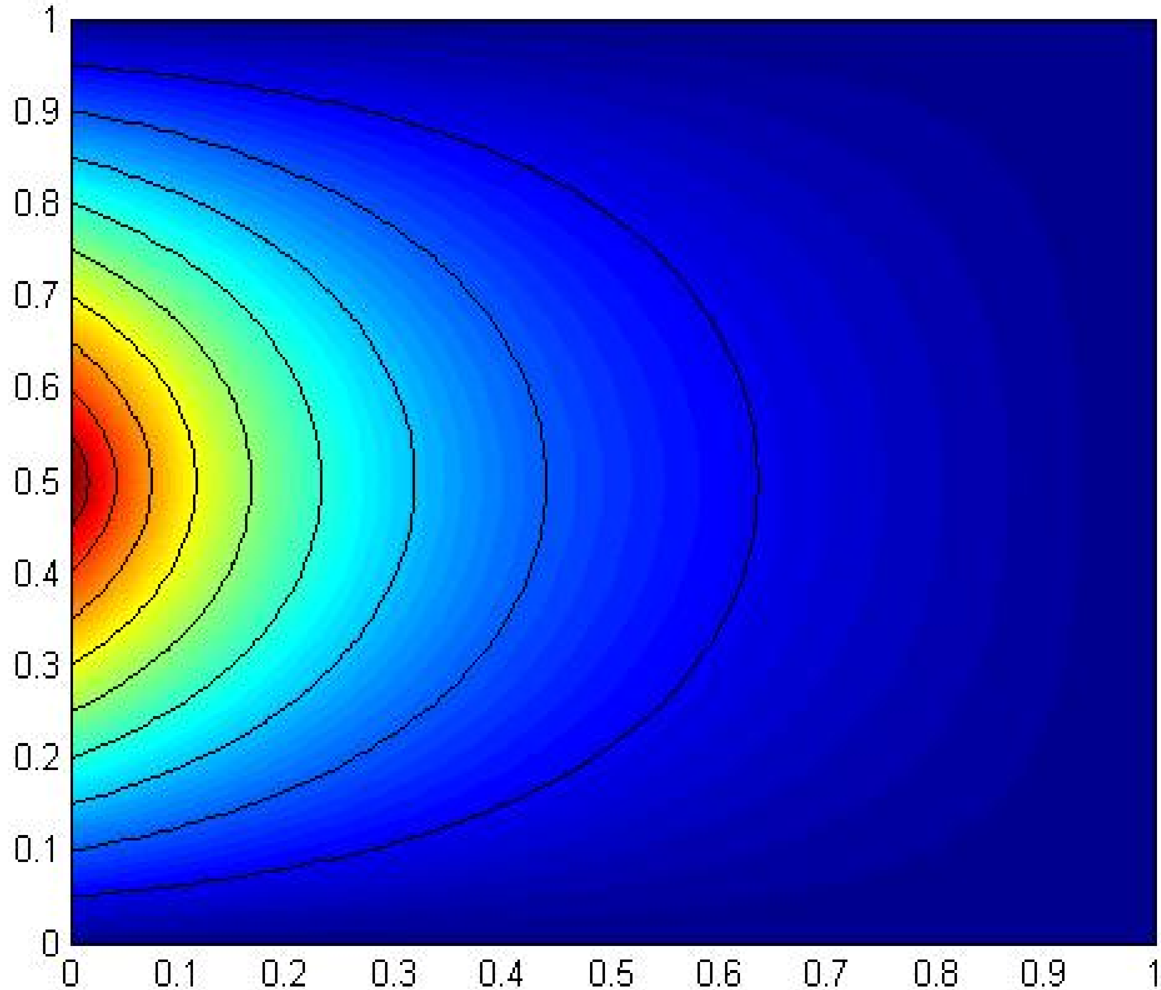
(See Slides #5 for a plot of $F(y)$ and related discussions.)

In this case,

$$a_m = \frac{2}{\sinh(m\pi)} \int_0^1 F(y) \sin(m\pi y)\, dy \ , \text{ and}$$

$$\begin{aligned} \int_0^1 F(y) \sin(m\pi y)\, dy &= \frac{2}{m^2 \pi^2} \ , \text{ if } m = 1, 5, 9, 13, \ldots \\ &= \frac{-2}{m^2 \pi^2} \ , \text{ if } m = 3, 7, 11, \ldots \\ &= 0\,, \quad \text{if } m \text{ is even}\,. \end{aligned} \tag{13}$$

Using the results in Eq. (13) and the infinite series in Eq. (9), we have the complete solution, as plotted in next page. (It uses 20 terms in the infinite series. Contour levels are 0.05, 0.1, 0.15, ..., 0.4, and 0.45. Red = hot; Blue = cold.)



0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1