Nonhomogeneous PDE - Heat equation with a forcing term

Example 1 Solve the PDE + boundary conditions

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + Q(x,t) \quad , \qquad \text{Eq. (1)}$$

(I) u(0,t) = 0
(II) u(1,t) = 0
(III) u(x,0) = P(x)

Strategy:
Step 1. Obtain the eigenfunctions in x, $G_n(x)$, that satisfy the PDE and boundary conditions (I) and (II)
Step 2. Expand u(x,t), Q(x,t), and P(x) in series of $G_n(x)$. This will convert the nonhomogeneous PDE to a set of simple nonhomogeneous ODEs.
Step 3. Solve the nonhomogeneous ODEs, use their solutions to reassemble the complete solution for the PDE

For the current example, our eigenfunctions are $G_n(x) = \sin(n\pi x)$, so we should try

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin(n\pi x) \quad , \qquad \text{Eq. (2)}$$

$$Q(x,t) = \sum_{n=1}^{\infty} q_n(t)\sin(n\pi x) \quad \Rightarrow \quad q_n(t) = 2\int_0^1 Q(x,t)\sin(n\pi x)\,dx \quad , \qquad \text{Eq. (3)}$$

$$P(x) = u(x,0) = \sum_{n=1}^{\infty} u_n(0)\sin(n\pi x) \quad \Rightarrow \quad u_n(0) = 2\int_0^1 P(x)\sin(n\pi x)\,dx \quad , \qquad \text{Eq. (4)}$$

From Eqs. (3) and (4), $q_n(t)$ and $u_n(0)$ have already been determined. Our task is to solve $u_n(t)$ and express it in $u_n(0)$ (the initial condition of $u_n(t)$) and $q_n(t)$ (the forcing that acts on $u_n(t)$).

Plugging Eq. (2) into the original PDE, we have

$$\frac{\partial}{\partial t}\sum_{n=1}^{\infty} u_n(t)\sin(n\pi x) = \frac{\partial^2}{\partial x^2}\sum_{n=1}^{\infty} u_n(t)\sin(n\pi x) + \sum_{n=1}^{\infty} q_n(t)\sin(n\pi x) \quad ,$$

$$=> \quad \sum_{n=1}^{\infty} \frac{d\,u_n(t)}{dt}\sin(n\pi x) = \sum_{n=1}^{\infty} -n^2\pi^2 u_n(t)\sin(n\pi x) + \sum_{n=1}^{\infty} q_n(t)\sin(n\pi x)$$

$$=> \quad \sum_{n=1}^{\infty}\left(\frac{d\,u_n(t)}{dt} + n^2\pi^2 u_n(t) - q_n(t)\right)\sin(n\pi x) = 0 \quad => \quad \frac{d\,u_n(t)}{dt} + n^2\pi^2 u_n(t) - q_n(t) = 0 \quad ,$$

or,

$$\frac{d\,u_n(t)}{dt} = -n^2\pi^2 u_n(t) + q_n(t) \quad , \quad n = 1, 2, 3, \ldots \qquad \text{Eq. (5)}$$

Equation (5) has the standard solution,

$$u_n(t) = u_n(0)\mathrm{e}^{-n^2\pi^2 t} + e^{-n^2\pi^2 t}\int_0^t q_n(t')\, e^{n^2\pi^2 t'}\, d\,t' \quad . \qquad \text{Eq. (6)}$$

Since $u_n(0)$ and $q_n(t)$ are known from Eqs. (3) and (4), we have the complete solution once the integral in Eq. (6) is evaluated to obtain $u_n(t)$; $u(x,t)$ can be evaluated by Eq. (2) once $u_n(t)$ is known.

Example 2: In example 1, find the solution for the case with

$$Q(x,t) = \sin(3\pi x)\, S(t),$$

where

$$S(t) = 1 \ , \ \text{if} \ 0 \le t \le T$$
$$= 0 \ , \ \text{if} \ t > T \ ,$$

and

$$P(x) = 5 \sin(2\pi x) + 2 \sin(3\pi x).$$

From Eqs. (3) and (4), we immediately obtain

$u_2(0) = 5$, $u_3(0) = 2$, and $u_n(0) = 0$ for all other n

$q_3(t) = S(t)$, and $q_n(t) = 0$ for all other n .

Thus, the expansion in Eq. (2) is reduced to just two terms,

$$u(x,t) = u_2(t) \sin(2\pi x) + u_3(t) \sin(3\pi x) \ , \qquad \text{Eq. (7)}$$

where

$$u_2(t) = u_2(0)\, \mathrm{e}^{-2^2\pi^2 t} = 5\, \mathrm{e}^{-4\pi^2 t} \quad , \qquad \text{Eq. (8)}$$

$$u_3(t) = u_3(0)\, \mathrm{e}^{-3^2\pi^2 t} + e^{-3^2\pi^2 t} \int_0^t q_3(t')\, e^{3^2\pi^2 t'}\, dt' \ . \qquad \text{Eq. (9)}$$

Case 1: Solution for t > T

For t > T, Eq. (9) will become

$$u_3(t) = u_3(0)\,\mathrm{e}^{-3^2\pi^2 t} + e^{-3^2\pi^2 t}\int_0^T e^{3^2\pi^2 t'}\,dt'$$

$$= 2\,\mathrm{e}^{-9\pi^2 t} + e^{-9\pi^2 t}\left(\frac{e^{9\pi^2 T}-1}{9\pi^2}\right) \;,$$

and the complete solution is

$$u(x,t) = 5\,\mathrm{e}^{-4\pi^2 t}\sin(2\pi x) + \left[2\,\mathrm{e}^{-9\pi^2 t} + e^{-9\pi^2 t}\left(\frac{e^{9\pi^2 T}-1}{9\pi^2}\right)\right]\sin(3\pi x) \;. \qquad \text{Eq. (10)}$$

Note #1: In this case, the solution decays to zero as $t \to \infty$

Note #2: In the absence of the forcing (setting Q(x,t) to zero), the solution is reduced to the familiar solution for the homogeneous heat equation,

$$u(x,t) = 5\,\mathrm{e}^{-4\pi^2 t}\sin(2\pi x) + 2\,\mathrm{e}^{-9\pi^2 t}\sin(3\pi x) \;.$$

Note #3: If the initial state is P(x) = 0, the solution is contributed entirely by the forcing:

$$u(x,t) = e^{-9\pi^2 t}\left(\frac{e^{9\pi^2 T}-1}{9\pi^2}\right)\sin(3\pi x) \;. \qquad \text{Eq. (11)}$$

Note #4: For the case with a very small T (i.e., "impulsive forcing"; the forcing Q is turned on at t = 0 then turned off after a very short amount of time), exp(αT) ≈ 1 + αT, so Eq. (11) can be approximated by

$$u(x,t) \approx e^{-9\pi^2 t} T \sin(3\pi x) \quad .$$

In this case, at the time when the forcing is switched off, i.e., at t = T, the system reaches an amplitude T. Afterward, it dacays exponentially just like the solution for the unforced heat equation.

Case 2: Solution for t < T

This is the case when the forcing is kept on for a long time (compared to the time, t, of our interest). If it is kept on forever, the equation might admit a nontrivial steady state solution depending on the forcing. In general, for t < T, Eq. (9) becomes

$$u_3(t) = u_3(0)\,\mathrm{e}^{-3^2\pi^2 t} + e^{-3^2\pi^2 t}\int_0^t e^{3^2\pi^2 t'}\,dt'$$
$$= 2\,\mathrm{e}^{-9\pi^2 t} + \left(\frac{1-e^{-9\pi^2 t}}{9\pi^2}\right) \quad ,$$

and the complete solution is

$$u(x,t) = 5\,\mathrm{e}^{-4\pi^2 t}\sin(2\pi x) + \left[2\,\mathrm{e}^{-9\pi^2 t} + \left(\frac{1-e^{-9\pi^2 t}}{9\pi^2}\right)\right]\sin(3\pi x) \quad . \qquad \text{Eq. (12)}$$

Notably, in this case a nontrivial steady state exists: $u(x,t) \to \frac{1}{9\pi^2}\sin(3\pi x)$ as t $\to\infty$ (and T $\to\infty$).

Exercise: In Example 2, what would be the behavior of the solution if the forcing is periodic in time. For example, if S(t) is replaced by sin(t) ?

Exercise: Note that the steady state solution in the preceding page can be readily obtained by setting $\partial u/\partial t$ to zero in the original PDE. Work out the detail and show that the result agrees with our conclusion at the bottom of the preceding page.