Revisit "solution to heat equation" in Slides #8

Recapitulation

Problem: For $u(x, t)$ defined on $x \in [0, 1]$ and $t \in [0, \infty)$, solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, with boundary conditions: (I) $u(0, t) = 0$, (II) $u(1, t) = 0$, (III) $u(x, 0) = 4\sin(3\pi x) + 7\sin(8\pi x)$

Solution:
Step 1: Separation of variable, $u(x, t) = G(x)H(t)$, leads to

$$\frac{1}{G}\frac{d^2 G}{d x^2} = c \quad , \quad G(0) = 0, G(1) = 0 \tag{1}$$

$$\frac{1}{H}\frac{d H}{d y} = c \quad . \tag{2}$$

Step 2: Equation (1) is an eigenvalue problem with eigenvalues, $c_n = -(k_n)^2$, where $k_n = n\pi$, $n = 1, 2, 3...$, and corresponding eigenfunctions $G_n(x) = \sin(k_n x)$. Using $c = c_n$ in Eq. (2) we obtains $H_n(t) = \exp(c_n t) = \exp(-(k_n)^2 t)$. Combining G and H we find

$$u_n(x, t) = G_n(x)\, H_n(t) = \sin(n\,\pi\,x)\,\exp(-n^2\pi^2 t)\ ,$$

which satisfies the PDE and b.c.'s (I) and (II). We are left with b.c. (III) (the "initial condition) to deal with.

Step 3: Since each $u_n(x, t)$, n = 1, 2, 3,..., satisfies the PDE + b.c.'s (I) & (II), the linear superposition of any combinations of them is also a solution. We therefore write the solution in its most general form,

$$u(x, t) = \sum_{n=1}^{\infty} a_n \, u_n(x, t) \quad , \qquad (3)$$

with yet undetermined coefficients, a_n.

Previous treatment (Slides #8):

At $t = 0$, Eq. (3) becomes

$$u(x, 0) = \sum_{n=1}^{\infty} a_n \, u_n(x, 0) = \sum_{n=1}^{\infty} a_n \sin(n \pi x) \quad . \qquad (4)$$

By comparing this expression with b.c. (III), we concluded that

$a_3 = 4$, $a_8 = 7$, and $a_n = 0$ for all other n .

The final solution is $u(x, t) = 4 \sin(3 \pi x) \exp(-9 \pi^2 t) + 7 \sin(8 \pi x) \exp(-64 \pi^2 t)$.

Discussion:

In the last step (in blue), we got it easy because the initial state in b.c. (III) happens to be a simple linear superposition of two of the eigenfunctions. The "visual comparison" approach works as long as the expression in Eq. (4) is unique. (In other words, for a given $u(x,0)$, there is a unique set of $\{a_n$, n = 1, 2, 3...$\}$ that satisfies the equation. This is actually true and is the reason that we got away with it.)

For a general initial state that does not resemble a superposition of a small (or even finite) number of the eigenfunctions, we need a systematic approach to solve the problem.

Formal derivation:

Consider a general form of the initial state in b.c. (III),

$$u(x, 0) = F(x) \,. \qquad \text{(III')}$$

Plugging it into Eq. (4) we have

$$F(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) \qquad (5)$$

Our goal is to obtain the unknown coefficients, a_n , from Eq. (5), for any given $F(x)$.

Step 1: Multiply an eigenfunction, sin($m\pi x$) to both sides of Eq. (5)

$$\sin(m\pi x)F(x) = \sin(m\pi x)\sum_{n=1}^{\infty} a_n \sin(n\pi x)$$

$$= \sum_{n=1}^{\infty} a_n \sin(m\pi x)\sin(n\pi x) \qquad (6)$$

Step 2: Integrate both sides of the equation from 0 to 1,

$$\int_0^1 \sin(m\pi x)F(x)\,dx = \int_0^1 \left[\sum_{n=1}^{\infty} a_n \sin(m\pi x)\sin(n\pi x)\right]dx$$

$$= \sum_{n=1}^{\infty} a_n \left[\int_0^1 \sin(m\pi x)\sin(n\pi x)\,dx\right] \qquad (7)$$

Step 3: Use the orthogonality relationship (see Slides #11),

$$\int_0^1 \sin(m\pi x)\sin(n\pi x)\,dx = 0,\ if\ m\neq n,\ \ \frac{1}{2},\ if\ m=n\ \ , \qquad (8)$$

we found that all terms in the r.h.s. of Eq. (7) are zero except when $n = m$.

This leads to the final expression of a_m ,

$$a_m = 2 \int_0^1 \sin(m\pi x) F(x) dx \quad . \tag{9}$$

Equation (9), combined with Eq. (3),

$$u(x, t) = \sum_{n=1}^{\infty} a_n u_n(x, t) \ ,$$

are our final solution to the whole problem.

Example 1: In the original problem in Slides #8, F(x) = 4 sin(3px) + 7 sin(8px),

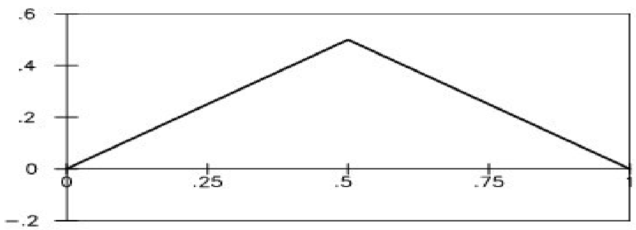
$$\Rightarrow \ a_m = 2 \int_0^1 \sin(m\pi x)(4\sin(3\pi x) + 7\sin(8\pi x)) dx \ .$$

Using the orthogonality relationship, Eq. (8), this immediately leads to the conclusion that a_3 ,= 4, a_8 = 7, and a_m = 0 for all other m.

Example 2: In the previous problem, consider the following initial state,

$$F(x) = x \qquad \text{for } 0 \le x \le 1/2 \ ,$$
$$F(x) = 1-x \qquad \text{for } 1/2 \le x \le 1 \ .$$

(This is a triangular-shaped distribution that does not resemble a linear superposition of a small or finite number of sinusoidal functions. See diagram below.)

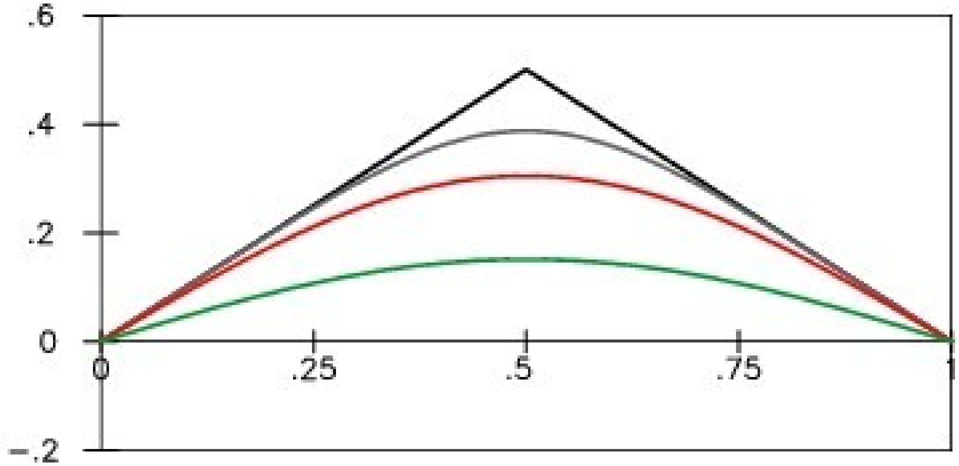


In this case, using Eq. (9), we have

$$a_m = 2\left[\int_0^{1/2} \sin(m\pi x)\, x\, dx + \int_{1/2}^{1} \sin(m\pi x)(1-x)\, dx\right] \ ,$$

$$\Rightarrow \quad a_m = \frac{4}{m^2\pi^2} \quad , m = 1, 5, 9, 13, \ldots$$
$$a_m = \frac{-4}{m^2\pi^2} \quad , m = 3, 7, 11, 15, \ldots$$
$$a_m = 0 \quad , \text{when } m \text{ is even} \ .$$

See next page for a plot of the solution.



Initial state (black);
Solution at $t = 0.01$ (gray), 0.03 (red), and 0.1 (green)

The solution is truncated at $n = 10$ for Eq. (3)