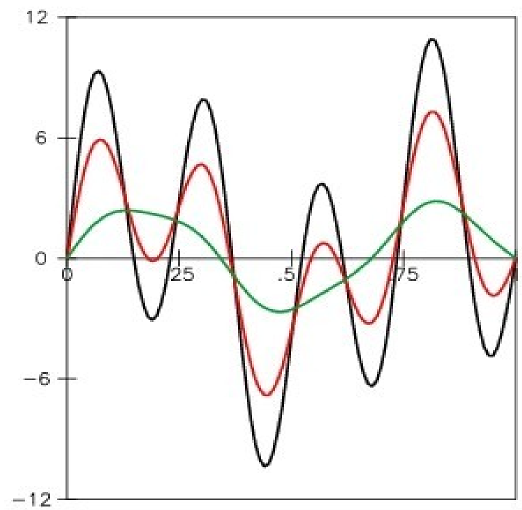
Some properties of heat (or "diffusion") equation , $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$

- Solution is "diffusive"; The sharper the temperature gradient is, the faster it is damped => Temperature profile becomes smoother as time increases

Example from Slides #4: Heat equation for $u(x, t)$ with b.c.'s (I) $u(0, t) = 0$, (II) $u(1, t) = 0$, (III) $u(x, 0) = 4\sin(3\pi x) + 7\sin(8\pi x)$



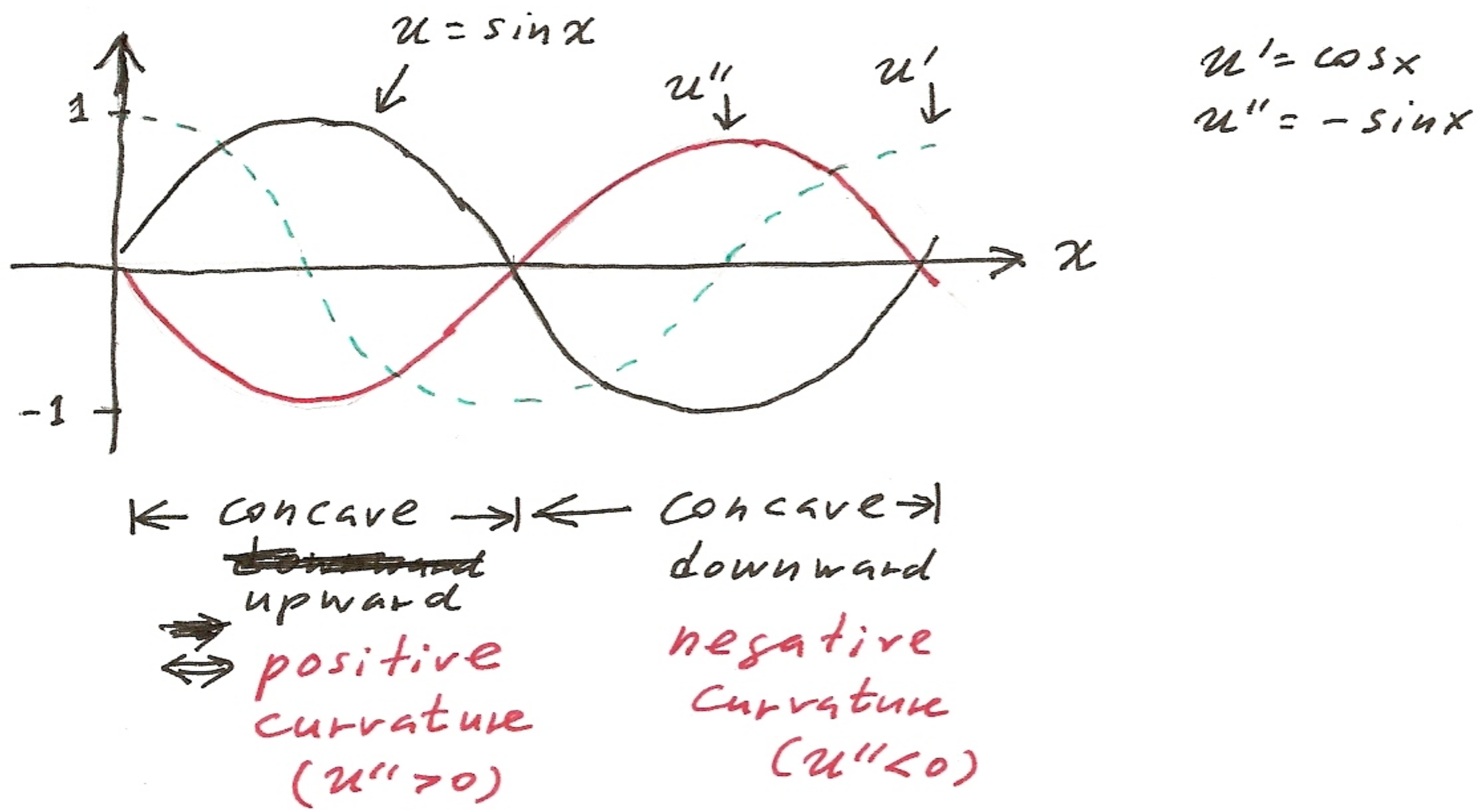
Solution: $u(x, t) = 4 \sin(3\pi x) \exp(-9\pi^2 t) + 7 \sin(8\pi x) \exp(-64\pi^2 t)$

$u(x,t)$ at $t = 0$ (black), 0.001 (red), and 0.005 (green)

We can understand the diffusive property of the heat equation by noting that the r. h. s. of the equation, $\partial^2 u/\partial x^2$, is the curvature (in x) of u for a given t.

Calculus: First derivative = slope   Second derivative = curvature

Example: $u(x) = \sin(x)$. For $0 < x < \pi$, the profile of u is concave downward
$\Leftrightarrow$ negative curvature, $u''(x) = -\sin(x) < 0$. For $\pi < x < 2\pi$ it's the opposite.

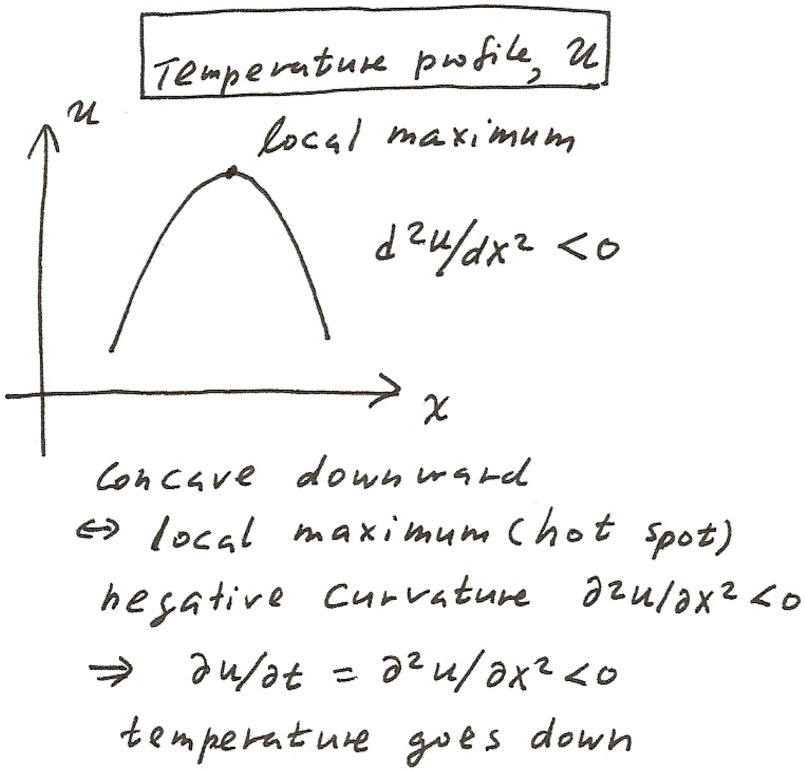


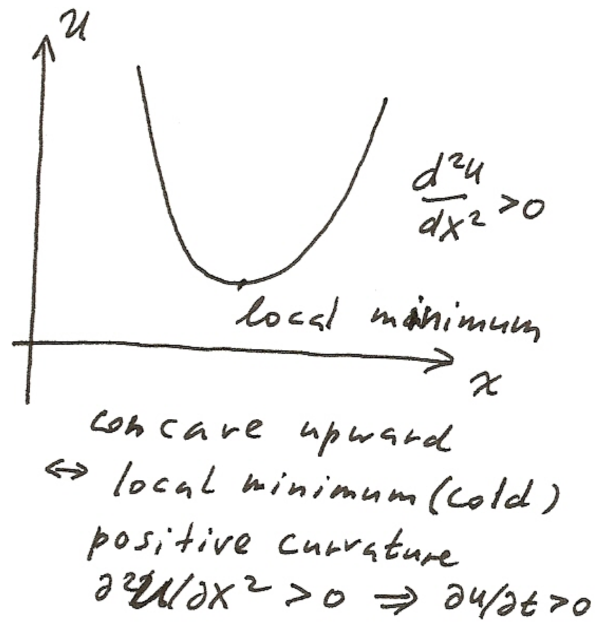
Heat equation, $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, in words: **The tendency of temperature ($\partial u/\partial t$) is proportional to the local curvature of the temperature profile ($\partial^2 u/\partial x^2$)**

Temperature profile $u(x, t)$ at a given t:

Concave downward (local maximum, hot spot; left diagram below)
$\Leftrightarrow \partial^2 u/\partial x^2 < 0$ (negative curvature) $\Leftrightarrow \partial u/\partial t < 0 \Leftrightarrow$ cools down

Concave upward (local minimum, cold spot; right diagram)
$\Leftrightarrow \partial^2 u/\partial x^2 > 0$ (positive curvature) $\Leftrightarrow \partial u/\partial t > 0 \Leftrightarrow$ warms up



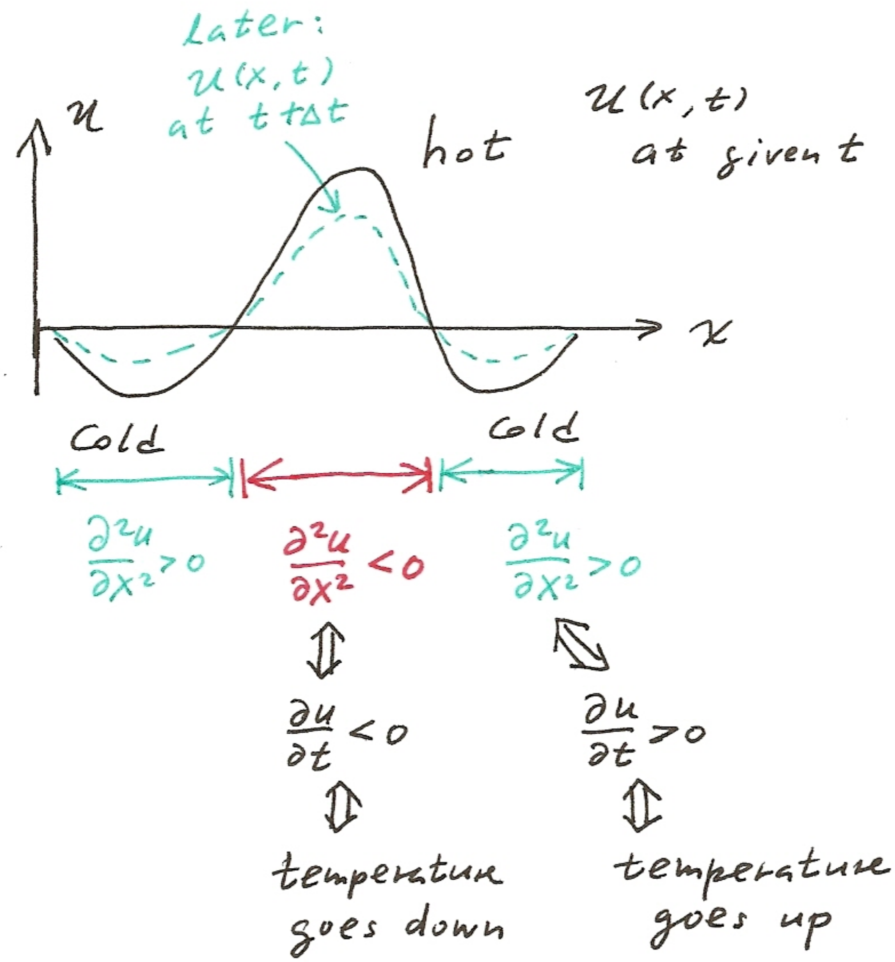


Process governed by heat equation:

Cooling down of hot spots; Warming up of cold spots

⇒Always a reduction of the contrast in temperature (temperature gradient)

⇒Temperature profile $u(x,t)$ becomes smoother as t increases



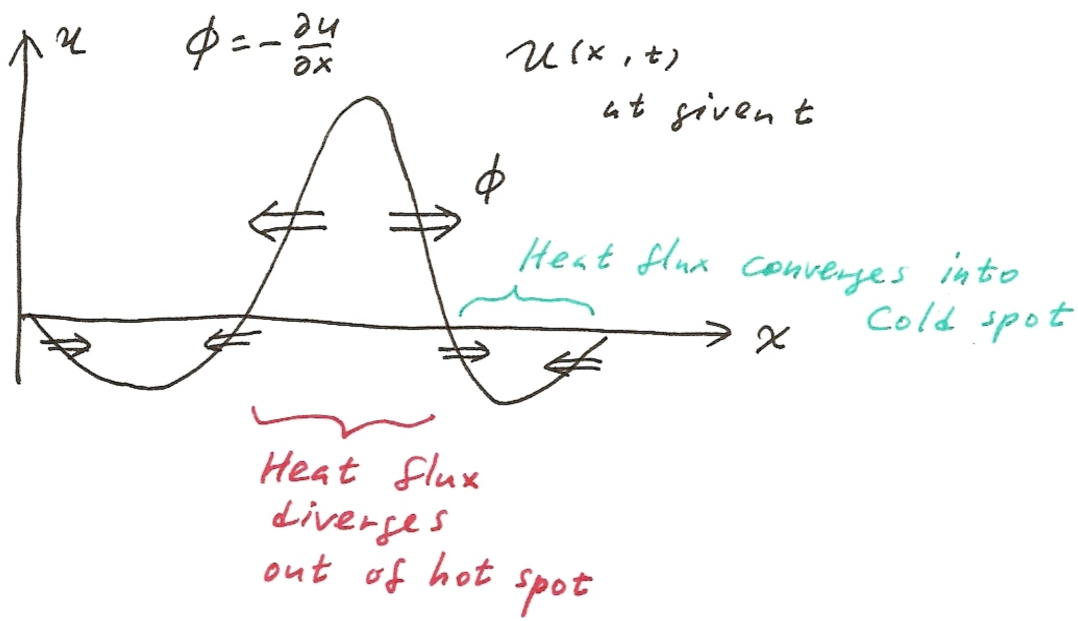
In terms of heat flux, $\phi \equiv -\ \partial u/\partial x$: Heat flux diverges out of the region with a negative curvature of temperature profile (where there is a temperature maximum; hot region) and diverges into the region with a positive curvature (where there is a temperature minimum; cold region)

Divergence of heat flux $\equiv \partial\phi/\partial x \equiv -\partial^2 u/\partial x^2$

(Recall that we define $\phi > 0$ when the flow of heat energy is toward the positive x direction)

Heat flux diverges $\Leftrightarrow \partial\phi/\partial x > 0 \Leftrightarrow \partial^2 u/\partial x^2 < 0 \Rightarrow \partial u/\partial t < 0 \Rightarrow$ temperature decreases

Heat flux converges $\Leftrightarrow \partial\phi/\partial x < 0 \Leftrightarrow \partial^2 u/\partial x^2 > 0 \Rightarrow \partial u/\partial t > 0 \Rightarrow$ temperature increases



Revisit the solution in the example in p.1 (detail in Slides #4):

Initial condition: $u(x, 0) = 4 \sin(3\pi\ x) + 7 \sin(8\pi\ x)$

Full solution: $u(x, t) = 4 \sin(3\pi\ x) \exp(-9\pi^2\ t) + 7 \sin(8\pi\ x) \exp(-64\pi^2\ t)$

The smoother component, $\sin(3\pi x)$, is damped at a slower rate ($\propto \exp(-9\pi^2\ t)$) compared to the less smooth component, $\sin(8\pi x)$. Although the initial amplitude of the latter is higher (7 vs. 4), after a while latter is almost entirely damped out.

At a large time, the solution is approximately $u(x, t) \approx 4 \sin(3\pi\ x) \exp(-9\pi^2\ t)$
This is what we see in the green curve in p. 1 of this set of slides.

The behavior of the solution described above is general.

Any solution to the heat equation must become smoother with time. (Save a few pathetic examples when heat flux is continuously pumped into the system though the boundaries, or when there is a persistent internal heat source without proper heat sink.)