A quick note on orthogonality relation

In Eq. (14) in Slides #4, we have

$$\sum_{n=1}^{\infty} a_n \sin(n\pi x) = F(x) \quad ,$$

where $F(x) = 4\sin(3\pi x) + 7\sin(8\pi x)$. How do we determine the coefficients, a_n, using the given information? In this simple example, a quick observation suffices for us to conclude that

$$a_3 = 4, \;\; a_8 = 7, \text{ and } a_n = 0 \text{ for all other } n\ .$$

However, if the F(x) has a more complicated form that does not resemble sin($n\pi$x), we will need a more systematic approach to obtain the a_n. The standard way to do so is to invoke the "orthogonality" property of the eigenfunctions. For $\{\sin(n\pi x)\}$, we have

$$\begin{aligned} A_{m,n} \equiv \int_0^1 \sin(n\pi x)\sin(m\pi x)\,dx &= 0 \quad , \text{ if } m \neq n \\ &= 1/2 \text{ , if } m = n \neq 0 \\ &= 0 \quad , \text{ if } m = n = 0 \end{aligned} \tag{1}$$

(Exercise: Verify that this is true.)

How to proceed:

Step 1: Multiply Eq. (14) in Slides #4 by $\sin(m\pi x)$

$$\sin(m\pi x)\sum_{n=1}^{\infty} a_n \sin(n\pi x) = \sin(m\pi x) F(x) \qquad (2)$$

Step 2: Integrate Eq. (2) over the whole domain, in our case from $x = 0$ to $x = 1$

$$\sum_{n=1}^{\infty} a_n \left(\int_0^1 \sin(m\pi x)\sin(n\pi x)\,dx \right) = \int_0^1 \sin(m\pi x) F(x)\,dx \quad ,$$

or,

$$\sum_{n=1}^{\infty} a_n A_{m,n} = \int_0^1 \sin(m\pi x) F(x)\,dx \quad , \qquad (3)$$

where the value of $A_{m,n}$ for given $\{m, n\}$ can be obtained from Eq. (1).

The fact that $A_{m,n}$ vanishes when $m \neq n$ (i.e., two distinctive eigenfunctions are "orthogonal" to each other - we will explain this later) is critical for our scheme to work. This leads to our last step,

Step 3:

For any given *m*, **by noting that** ***$A_{m,n}$*** **vanishes when** ***$m \neq n$*** **, the infinite sum in the left hand side of Eq. (3) is reduced to a single term**

$$
\begin{aligned}
\sum_{n=1}^{\infty} a_n A_{m,n} &= a_1 A_{m,1} + a_2 A_{m,2} + a_3 A_{m,3} + \ldots + a_m A_{m,m} + \ldots \\
&= 0 + 0 + 0 + \ldots + a_m A_{m,m} + 0 + \ldots \\
&= a_m A_{m,m} \\
&= a_m/2 \qquad (\text{since } A_{m,m} = 1/2, \text{ from Eq. (1)}) \qquad (4)
\end{aligned}
$$

Using (3) and (4), we obtain a useful expression for the coefficients a_m ,

$$a_m = 2\int_0^1 \sin(m\pi x) F(x)\, dx \quad \text{, for a non-zero } m \qquad (5)$$

We are done! This is all we need to evaluate the coefficients from the $F(x)$ in the initial condition.

Example 1

Let's get back to the example in Slides #4, where $F(x) = 4\sin(3\pi x) + 7\sin(8\pi x)$. Using Eq. (5), we have

$$\begin{aligned} a_m &= 2\int_0^1 \sin(m\pi x) F(x)\, dx \\ &= 2\int_0^1 \sin(m\pi x)\left(4\sin(3\pi x) + 7\sin(8\pi x)\right) dx \\ &= 2\,(4\ A_{m,3} + 7\ A_{m,8}\ ) \end{aligned}$$

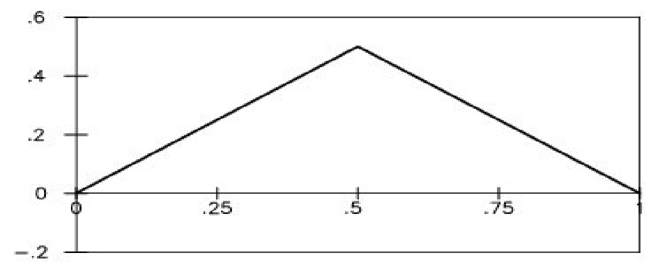
So, we have $a_3 = 4$, $a_8 = 7$, and $a_n = 0$ for all other n.

Example 2

Let's try a more complicated initial condition for the Heat equation in Slides #4:

(iii) $u(x, 0) = F(x)$, where $F(x) = x$, $0 \le x \le 1/2$
$= 1 - x$, $1/2 \le x \le 1$

(This is a triangular-shaped distribution. See diagram below.)



In this case, using Eq. (5), we have

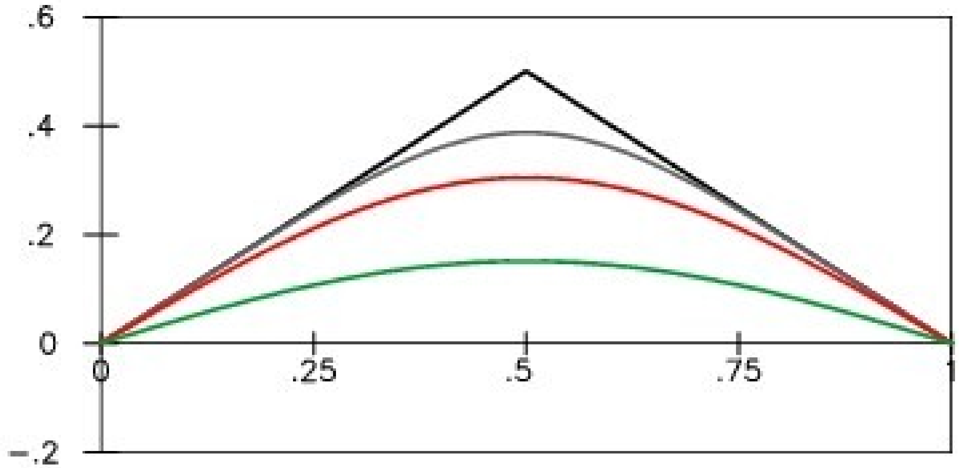
$$a_m = 2\left[\int_0^{1/2} \sin(m\pi x)\, x\, dx + \int_{1/2}^{1} \sin(m\pi x)(1-x)\, dx\right] ,$$

$$\Rightarrow a_m = \frac{4}{m^2\pi^2} \quad , m = 1, 5, 9, 13, \ldots$$

$$a_m = \frac{-4}{m^2\pi^2} \quad , m = 3, 7, 11, 15, \ldots$$

$$a_m = 0 \quad , \text{when } m \text{ is even} .$$

Plugging the a_m back to the solution of the Heat equation in Slides #6, we obtain the complete solution under the new initial state. The following is a plot of the solution $u(x,t)$ at selected values of t.



Initial state (black);
Solution at $t = 0.01$ (gray), 0.03 (red), and 0.1 (green)

The solution is truncated at $n = 10$