

Summary of Chapter 5

(Why we have orthogonal eigenfunctions for so many physical problems)

Key: A Sturm-Liouville problem has orthogonal eigenfunctions

Sturm-Liouville (eigenvalue) problem:

$$\frac{d}{dx} \left[P(x) \frac{du}{dx} \right] + Q(x)u - \lambda R(x)u = 0, \quad (1)$$

for $u(x)$ defined on $x \in [a, b]$, plus homogeneous b.c.'s

$$A u(a) + B u'(a) = 0 \quad (u' \equiv du/dx) \quad (\text{I})$$

$$C u(b) + D u'(b) = 0. \quad (\text{II})$$

Remarks:

- (1) The forms of the ODE and b.c.'s above are general enough that many physical problems can be converted to a standard Sturm-Liouville problem => Orthogonality of eigenfunctions
- (2) It is crucial that the b.c.'s are homogeneous. If they are not, there may not be orthogonal eigenfunctions for the system.

Proof of orthogonality...

Step 1: Define the operator L as

$$L\{u\} \equiv \frac{d}{dx} \left[P(x) \frac{du}{dx} \right] + Q(x)u ,$$

such that the original Eq. (1) in the Sturm-Liouville system can be written as

$$L\{u\} = \lambda R(x) u \tag{2}$$

Let u and v be two solutions (need not be eigenfunctions at this point) to the Sturm-Liouville problem, then

$$\begin{aligned} \int_a^b v L\{u\} - u L\{v\} dx &= \int_a^b v \left\{ \frac{d}{dx} \left[P \frac{du}{dx} \right] + Qu \right\} - u \left\{ \frac{d}{dx} \left[P \frac{dv}{dx} \right] + Qv \right\} dx \\ &= \int_a^b v \frac{d}{dx} \left[P \frac{du}{dx} \right] - u \frac{d}{dx} \left[P \frac{dv}{dx} \right] dx \\ &= \int_a^b \frac{d}{dx} \left[P \left(v \frac{du}{dx} - u \frac{dv}{dx} \right) \right] dx \\ &= \left[P \left(v \frac{du}{dx} - u \frac{dv}{dx} \right) \right]_a^b \\ &= P(b) \left[v(b)u'(b) - u(b)v'(b) \right] - P(a) \left[v(a)u'(a) - u(a)v'(a) \right] \\ &= 0 \end{aligned} \tag{3}$$

See next page for an explanation why the green-colored expression is identically zero.

[Note: There was a typo in my blackboard derivation (April 8 lecture) for the above equation. I accidentally put Qdu/dx in the place of Qu . Please correct your notes. -- HPH]

(Addendum to the derivation in previous page)

Since u and v are two solutions to the Sturm-Liouville system, they both satisfy the b.c.'s (I) and (II),

$$A u(a) + B u'(a) = 0 \quad (\text{I-u})$$

$$C u(b) + D u'(b) = 0 . \quad (\text{II-u})$$

$$A v(a) + B v'(a) = 0 \quad (\text{I-v})$$

$$C v(b) + D v'(b) = 0 . \quad (\text{II-v})$$

From (I-u) and (I-v), we have

$$u(a) = (-B/A) u'(a) , \text{ and } v'(a) = (-A/B) v(a) \Rightarrow v(a)u'(a) - u(a)v'(a) = 0$$

similarly, from (II-u) and (II-v) we can establish that $v(b)u'(b) - u(b)v'(b) = 0$.

Step 2: Now, consider that u and v are two eigenfunctions, $u = \phi_m$, $v = \phi_n$, of the Sturm-Liouville problem corresponding to eigenvalues λ_m and λ_n . Then,

$$L\{\phi_m\} = \lambda_m R(x) \phi_m \quad , \quad L\{\phi_n\} = \lambda_n R(x) \phi_n \quad .$$

Using Eq. (3), we have

$$0 = \int_a^b \phi_n L\{\phi_m\} - \phi_m L\{\phi_n\} dx = \int_a^b (\lambda_m - \lambda_n) \phi_m \phi_n R(x) dx \quad .$$

Therefore, as long as $\lambda_m \neq \lambda_n$, we have the orthogonal relationship

$$\int_a^b \phi_m \phi_n R(x) dx = 0 \quad .$$

Remarks:

In addition to orthogonality of eigenfunctions, it can be shown that

- The eigenvalues of the Sturm-Liouville system are real and have a lower bound (but no upper bound); The eigenvalues can be ordered as $\lambda_1 < \lambda_2 < \lambda_3 < \dots$, with λ_1 the smallest eigenvalue
- There is a one-to-one correspondence between an eigenvalue and an eigenfunction
- The eigenfunctions form a complete basis for piece-wise continuous functions defined on $[a, b]$, meaning that any function $f(x)$ that is piece-wise continuous can be represented by the eigenfunction expansion,

$$f(x) \approx \sum_n a_n \phi_n .$$

See p. 163 in textbook for further detail.