Summary of Chapter 5
(When do we have orthogonal eigenfunctions for our boundary value problem?)

Key: A **Sturm-Lioville problem** has orthogonal eigenfunctions

Sturm-Liouville (eigenvalue) problem:

$$\frac{d}{dx}\left[P(x)\frac{du}{dx}\right] + Q(x)u - \lambda R(x)u = 0 \quad , \qquad (1)$$

for $u(x)$ defined on $x \in [a, b]$, plus homogeneous b.c.'s

$$A\, u(a) + B\, u'(a) = 0 \quad (u' \equiv du/dx) \qquad (I)$$
$$C\, u(b) + D\, u'(b) = 0\ . \qquad (II)$$

Remarks:

(1) The forms of the ODE and b.c.'s above are general enough that many physical problems can be converted to a standard Sturm-Liouville problem => Orthogonality of eigenfunctions

(2) It is crucial that the b.c.'s are homogeneous. If they are not, there may not be orthogonal eigenfunctions for the system.

Proof of orthogonality...

Step 1: Define the operator ***L*** as

$$\boldsymbol{L}\{u\} \equiv \frac{d}{dx}\left[P(x)\frac{du}{dx}\right] + Q(x)u \ ,$$

such that the original Eq. (1) in the Sturm-Liouville system can be written as

$$\boldsymbol{L}\{u\} = \lambda\, R(x)\, u \qquad (2)$$

Let *u* and *v* be two solutions (need not be eigenfunctions at this point) to the Sturm-Liouville problem, then

$$\begin{aligned}
\int_a^b v\boldsymbol{L}\{u\} - u\boldsymbol{L}\{v\}\,dx &= \int_a^b v\{\frac{d}{dx}[P\frac{du}{dx}] + Qu\} - u\{\frac{d}{dx}[P\frac{dv}{dx}] + Qv\}\ dx \\
&= \int_a^b v\frac{d}{dx}[P\frac{du}{dx}] - u\frac{d}{dx}[P\frac{dv}{dx}]\ dx \\
&= \int_a^b \frac{d}{dx}[P(v\frac{du}{dx} - u\frac{dv}{dx})]\ dx \\
&= \left[P(v\frac{du}{dx} - u\frac{dv}{dx})\right]_a^b \\
&= \mathrm{P}(b)[v(b)u'(b) - u(b)v'(b)] - \mathrm{P}(a)[v(a)u'(a) - u(a)v'(a)] \\
&= 0 \qquad (3)
\end{aligned}$$

See next page for an explanation why the green-colored expression is identically zero.

(Addendum to the derivation in previous page)

Since u and v are two solutions to the Sturm-Liouville system, they both satisfy the b.c.'s (I) and (II),

$$A\,u(a) + B\,u'(a) = 0 \qquad \text{(I-u)} \qquad\qquad A\,v(a) + B\,v'(a) = 0 \qquad \text{(I-v)}$$
$$C\,u(b) + D\,u'(b) = 0\,. \qquad \text{(II-u)} \qquad\qquad C\,v(b) + D\,v'(b) = 0\,. \qquad \text{(II-v)}$$

From (I-u) and (I-v), we have

$$u(a) = (-B/A)\,u'(a)\,, \;\text{ and }\; v'(a) = (-A/B)\,v(a) \;\Rightarrow\; v(a)u'(a) - u(a)v'(a) = 0$$

similarly, from (II-u) and (II-v) we can establish that $v(b)u'(b) - u(b)v'(b) = 0$.

Step 2: Now, consider that u and v are two eigenfunctions, $u = \phi_m$, $v = \phi_n$, of the Sturm-Liouville problem corresponding to eigenvalues λ_m and λ_n. Then,

$$\boldsymbol{L}\{\phi_m\} = \lambda_m R(x)\,\phi_m \quad , \qquad \boldsymbol{L}\{\phi_n\} = \lambda_n R(x)\,\phi_n \ .$$

Using Eq. (3), we have

$$0 = \int_a^b \phi_n \boldsymbol{L}\{\phi_m\} - \phi_m \boldsymbol{L}\{\phi_n\}\, dx = \int_a^b (\lambda_m - \lambda_n)\,\phi_m \phi_n R(x)\, dx \ .$$

Therefore, as long as $\lambda_m \neq \lambda_n$, we have the orthogonality relation

$$\int_a^b \phi_m \phi_n R(x)\, dx = 0 \ .$$

Remarks:

In addition to orthogonality of eigenfunctions, it can be shown that

- The eigenvalues of the Sturm-Liouville system are discrete and real, and they have a lower bound (but no upper bound); The eigenvalues can be ordered as $\lambda_1 < \lambda_2 < \lambda_3 < \ldots$, with λ_1 the smallest eigenvalue

- There is a one-to-one correspondence between an eigenvalue and an eigenfunction

- The eigenfunctions form a complete basis for piece-wise continuous functions defined on $[a, b]$, meaning that any function $f(x)$ that is piece-wise continuous can be represented by the eigenfunction expansion,

$$f(x) \approx \sum_n a_n \phi_n \quad .$$

See p. 163 in textbook for further detail.