# Lecture 2



$$\begin{array}{ccc}
(*)\ \text{difficult math eq.} & \longrightarrow & \text{simpler eqs.} \\
 & & \Big\downarrow \text{solve} \\
\text{full sol. for } (*) & \xleftarrow{\ \#\ } & \text{solutions}
\end{array}$$

full sol. $= \sum$ building blocks

Example (ODE)

$u(x)$

$$\boxed{\frac{du}{dx} = u \qquad u(0) = 1}$$

b.c.

Calculus : (power series)

✓ $u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$

$\Downarrow$

$\{x^N\}$

$u'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots$

H.S.A.

$a_1 = a_0$

$2a_2 = a_1$

$3a_3 = a_2$

$4a_4 = a_3$

$\vdots$

$$a_1 = a_0 = \frac{a_0}{1!}$$

$$a_2 = \frac{a_1}{2} = \frac{a_0}{2} = \frac{a_0}{2!}$$

$$a_3 = \frac{a_2}{3} = \frac{a_0}{3 \cdot 2} = \frac{a_0}{3!}$$

$$a_4 = \dots = \frac{a_0}{4 \cdot 3 \cdot 2} = \frac{a_0}{4!}$$

$\vdots$

$$a_N = \dots = \frac{a_0}{N!}$$

$$u(x) = a_0 \left[ 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \right]$$

apply b.c. $(u(0)=1) \Longrightarrow 1 = a_0 [1 + 0 + 0 \dots]$

$\Rightarrow a_0 = 1$

Full sol. :

$$u(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (0! \equiv 1)$$

$\underline{u(x) = e^x}$

$u(0.2) = ?$

$$u(0.2) = \underline{\underline{1 + 0.2}} + \frac{(0.2)(0.2)}{2} + \frac{(0.2)(0.2)(0.2)}{3 \cdot 2} + \dots$$

$= 1$
$1.2$
$1.22$
$1.221\dots$

Exp(2)

$\boxed{0}\,\boxed{.}\,\boxed{2}$ $\boxed{\text{Exp}}$ enter

1.221402...

$$\frac{du}{dx} = u \qquad u(0) = 1$$

$$\frac{d}{dx} u = u$$

$$u(x) = \underbrace{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots}$$

Can evaluate $u(x)$

Exp

$e^x$

$e^{\alpha x}$

PDE $\longrightarrow$ ODE

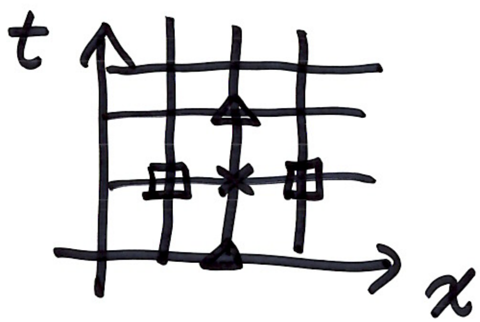
$$\frac{\partial u}{\partial t} = \boxed{\frac{\partial u}{\partial x}} \rightarrow \underline{\text{Const}}$$

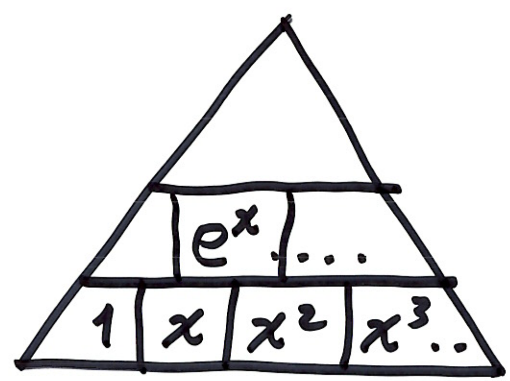
$$\underline{u(x,t)} = \sum a_n(t) G_n(x)$$

$$\frac{\partial u}{\partial t} = \left(\frac{\partial u}{\partial t}\right)_x$$

$$\frac{\partial u}{\partial x} = \left(\frac{\partial u}{\partial x}\right)_t$$

$\underline{u(x,t)}$



✓ $\dfrac{du}{dx} = u$ $\qquad u = \underline{\underline{e^{x}}} = \left(1 + x + \dfrac{x^2}{2!} + \dots\right)$

$\underline{\dfrac{du}{dx} = c\,u}$ (c is a const)

change of var $\hat{x} \equiv cx$

$\dfrac{du}{d\hat{x}} = u \Rightarrow u = e^{\hat{x}} = \underline{e^{cx}}$

$\left(1 + cx + \dfrac{(cx)^2}{2!} + \dots\right)$

$\dfrac{d}{dx}\,u = c\,u$

$\underline{\underline{\mathcal{L}\,u}} = \underline{\underline{c\,u}}$

Ex: (2nd-order ODE)

$u(x)$ $$\frac{d^2u}{dx^2} - 3\frac{du}{dx} + 2u = 0 \quad + \text{ 2 b.c.'s}$$

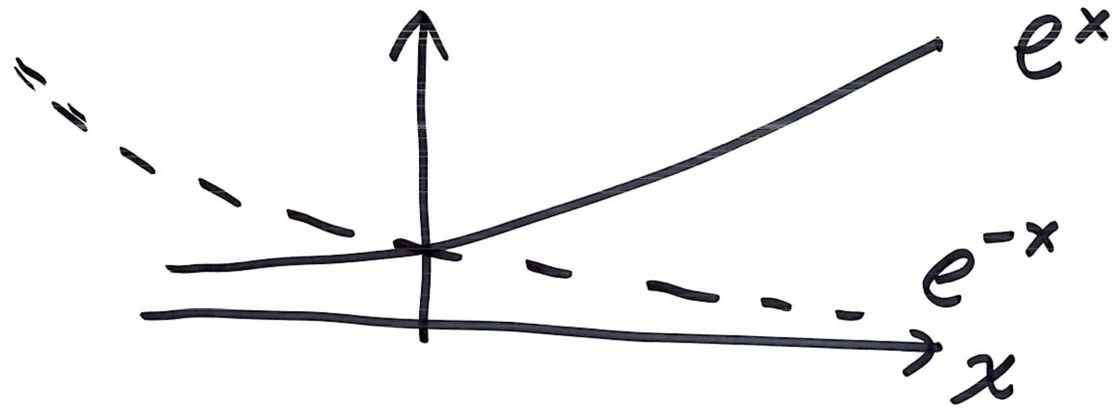
$$u \sim \underline{e^{\alpha x}}$$

$$\alpha^2 \underline{e^{\alpha x}} - 3\alpha \underline{e^{\alpha x}} + 2 \underline{e^{\alpha x}} = 0$$

$$\Rightarrow \quad \alpha^2 - 3\alpha + 2 = 0$$

$$\Rightarrow \quad \alpha = 1, 2$$

$$u(x) = \underline{\underline{A}}\, e^{1x} + \underline{\underline{B}}\, e^{2x}$$




often not good fit

PDE + $\boxed{\text{boundary conditions}}$

More popular (and useful) building blocks come from the solutions of

$$\frac{d^2 u}{dx^2} = c\,u$$

Ex: $u(x)$

$$\frac{d^2u}{dx^2} = u \quad + \quad 2 \ \text{b.c.'s}$$

$e^x$

$\{e^x, e^{-x}\}$

$$u(x) = a_0 + a_1 x + a_2 x^2 + \dots$$

$$u'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots$$

$$u''(x) = 2a_2 + (3\cdot2)a_3 x + (4\cdot3)a_4 x^2 + (5\cdot4)a_5 x^3 + \dots$$

$$2a_2 = a_0$$
$$(4\cdot3)a_4 = a_2$$
$$(6\cdot5)a_6 = a_4$$
$$\vdots$$

$$(3\cdot2)a_3 = a_1$$
$$(5\cdot4)a_5 = a_3$$
$$\vdots$$

$$\frac{e^x + e^{-x}}{2} = \cosh(x)$$

sol. 1:

$$u_1(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots \quad \cosh(x)$$

$$u_2(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \quad \sinh(x) = \frac{e^x - e^{-x}}{2}$$

---

Repeat but for $\frac{d^2u}{dx^2} = -u$

$$u_1(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \quad \cos(x)$$

$$u_2(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \quad \sin(x)$$