Lecture 3

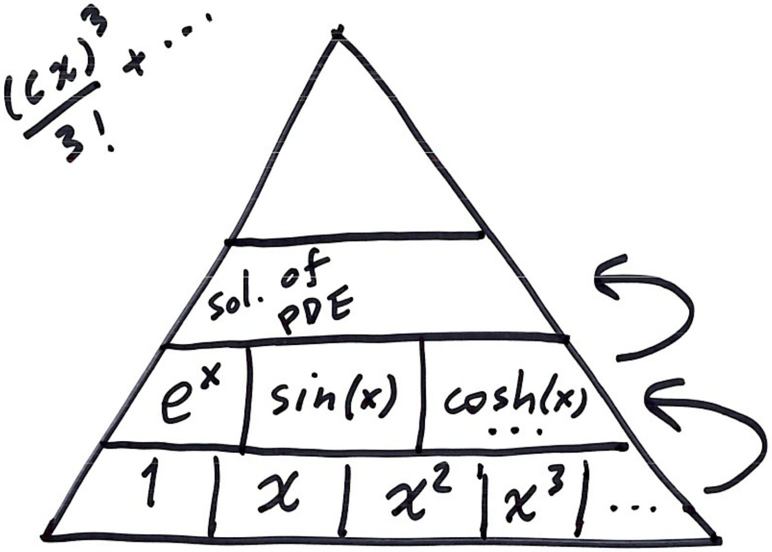
Recap:

* building blocks
* special

e.g. $e^{cx} = 1 + cx + \frac{(cx)^2}{2!} + \frac{(cx)^3}{3!} + \dots$ $\ast$

$u(x)$

sol. of $\frac{d}{dx} u = c\, u$



Lec 1, 2

$$\frac{\partial u}{\partial t} = \boxed{\frac{\partial u}{\partial x}}$$ PDE

$u(x,t)$

c

$$\frac{du}{dt} = c\,u$$ ODE

Lec 2

$$\frac{d^2u}{dx^2} = u \quad \text{(*)} \quad \rightarrow \quad \begin{cases} u(x) = \underline{\cosh(x)} = \left[1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right] \\ u(x) = \sinh(x) \end{cases}$$

$$\frac{d^2u}{dx^2} = -u \quad \text{(**)} \quad \rightarrow \quad \begin{cases} u(x) = \cos(x) \\ u(x) = \underline{\sin(x)} = \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right] \end{cases}$$

$$\frac{d^2}{dx^2} u = c\, u \qquad \underline{c > 0} \quad \text{positive const}$$

$$\underline{\hat{x} \equiv \sqrt{c}\, x}$$

like (*) :
$$\frac{d^2}{d\hat{x}^2} u = u \Rightarrow u = \begin{cases} \cosh(\hat{x}) = \cosh(\sqrt{c}\, x) \\ \sinh(\hat{x}) = \sinh(\sqrt{c}\, x) \end{cases}$$

$$f(x) = \cosh(3x)$$
$$f'(x) = 3 \cdot \sinh(3x)$$
$$f''(x) = 3 \cdot 3 \cdot \cosh(3x) = 3^2 \cosh(3x)$$

$$\frac{d^2}{dx^2} u = c\, u \qquad \underline{c<0} \quad \text{negative const}$$

$$\hat{x} \equiv \sqrt{-c}\, x \qquad \boxed{\left(\sqrt{-c}\right)^2 = -c > 0}$$

like (**) : $\dfrac{d^2}{d\hat{x}^2} u = -u$

$$\Rightarrow \quad u = \begin{cases} \cos(\hat{x}) = \cos(\sqrt{-c}\, x) \\ \sin(\hat{x}) = \sin(\sqrt{-c}\, x) \end{cases}$$

$$f(x) = \sin(3x)$$

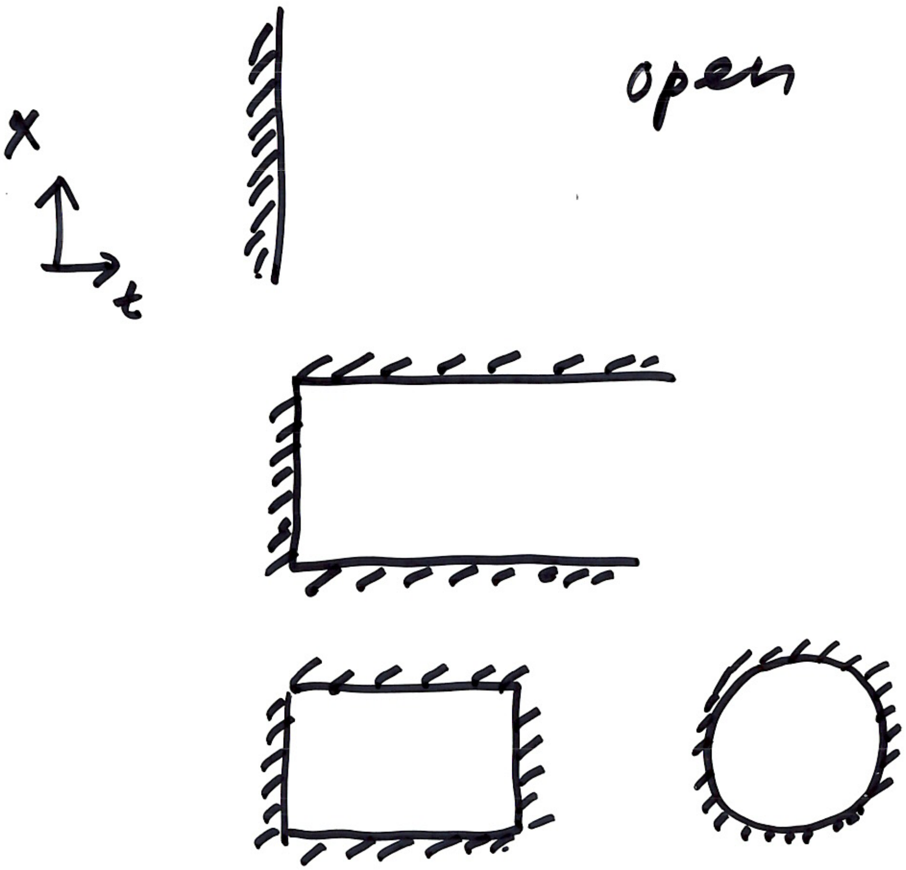
$$f'(x) = 3\cos(3x)$$

$$f''(x) = -3 \cdot 3 \sin(3x)$$

$$= \underbrace{-3^2}_{c} \sin(\underset{\sqrt{-c}}{\underline{\underline{3}}}x)$$

Boundary Conditions

PDE (2 var) (x, t)



ODE $u(\underline{\underline{x}})$

1st-order ODE $\underline{u(a) = \square}$

2nd-order ODE

$u(a) = \square$

$u'(a) = \square$

initial cond.

closed b.c.

$x = a$ $\quad$ $x = b$

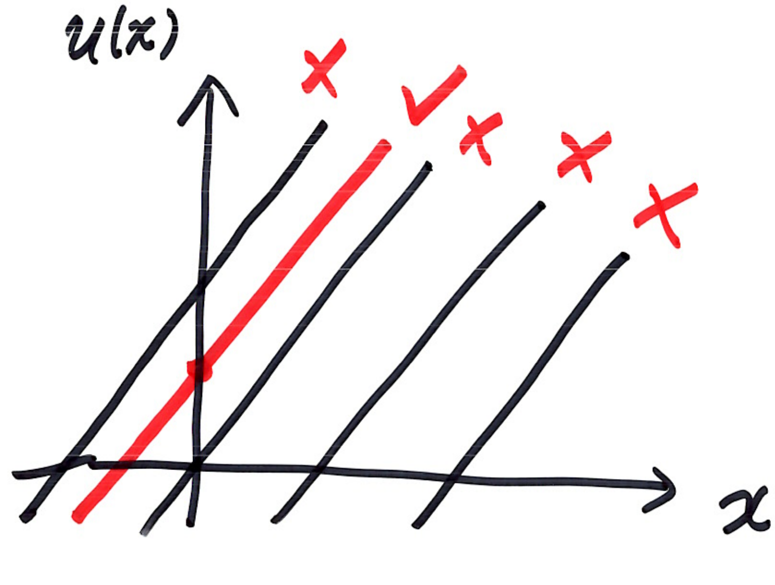
$u(a) = \square$ $\quad$ $u(b) = \square$

Simple examples (ODE)

Ex: 1st-order ODE

$u \equiv u(x)$

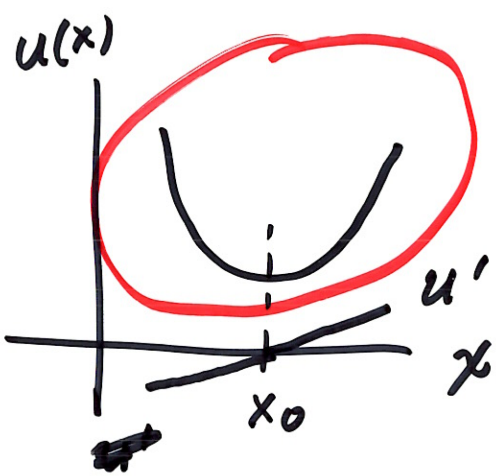
$$\begin{cases} \dfrac{du}{dx} = 2 \\ u(0) = 1 \end{cases}$$



Ex: 2nd-order ODE

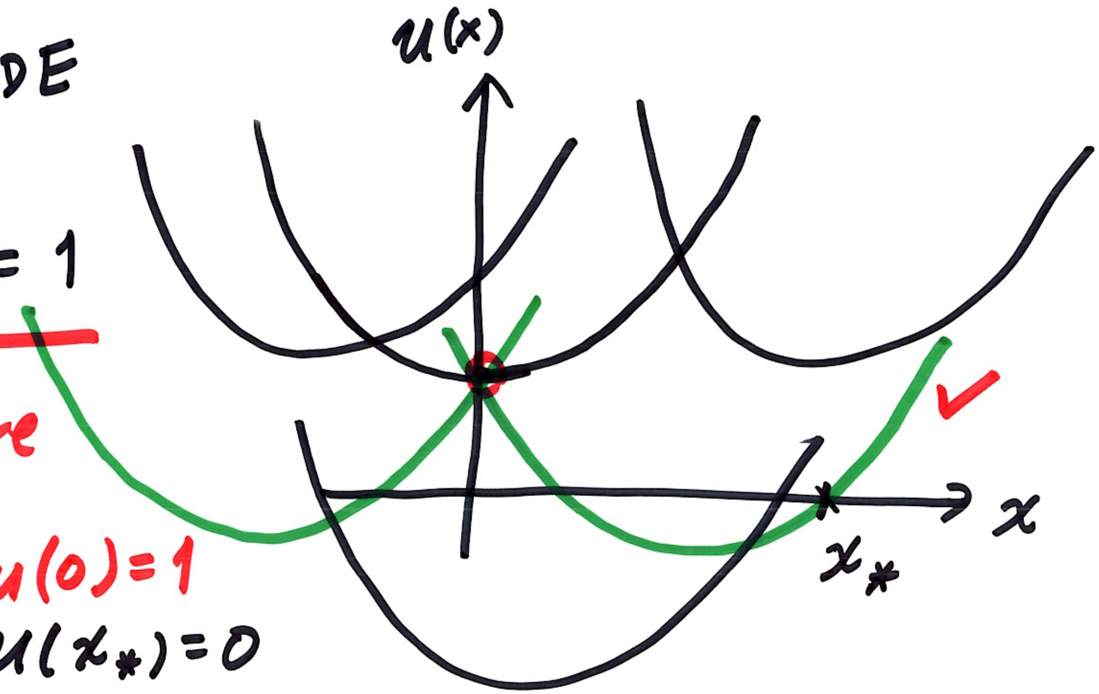
$u \equiv u(x)$

$$\frac{d^2u}{dx^2} = 1$$



positive u''

$$\begin{cases} u(0) = 1 \\ u(x_*) = 0 \end{cases}$$



PDE → classical linear PDEs

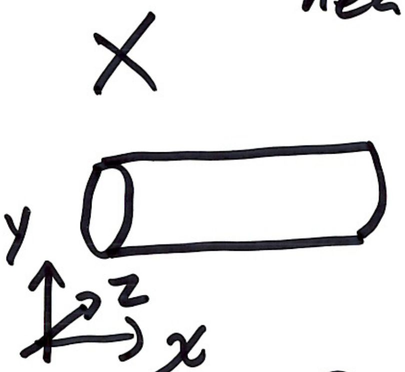
- Heat eq. ✓
- Laplace's eq. ✓
- Wave eq. ✓

✓ * end-to-end solution

* effect of b.c. (uniqueness/existence of sol.)

"1-D Heat eq." ("Diffusion" eq.)

heat transfer along a thin metal rod

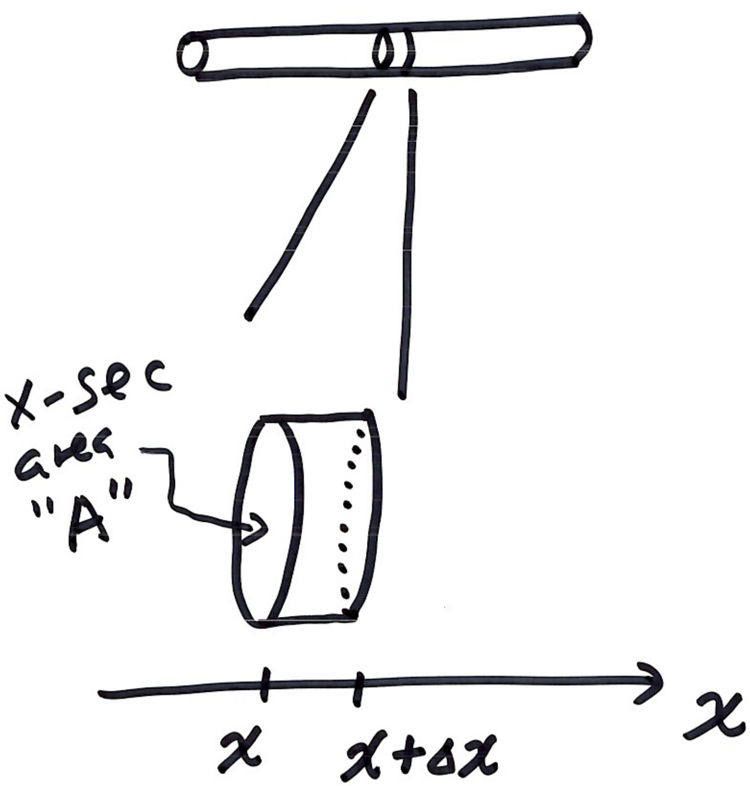


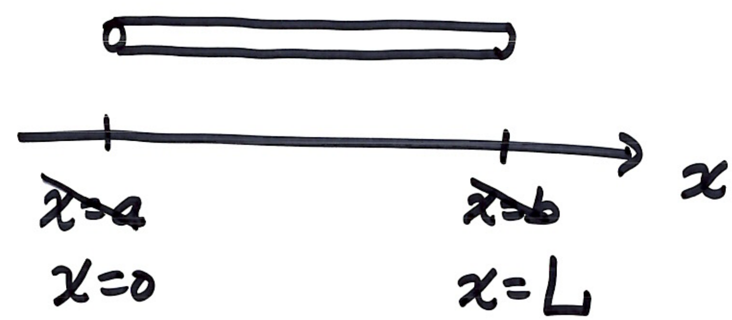
3-D

u ← temperature

$u(x, t)$ ← time

also need: side boundaries are thermally insulated local <u>energy</u> budget → static & u





$0 \le x \le L$

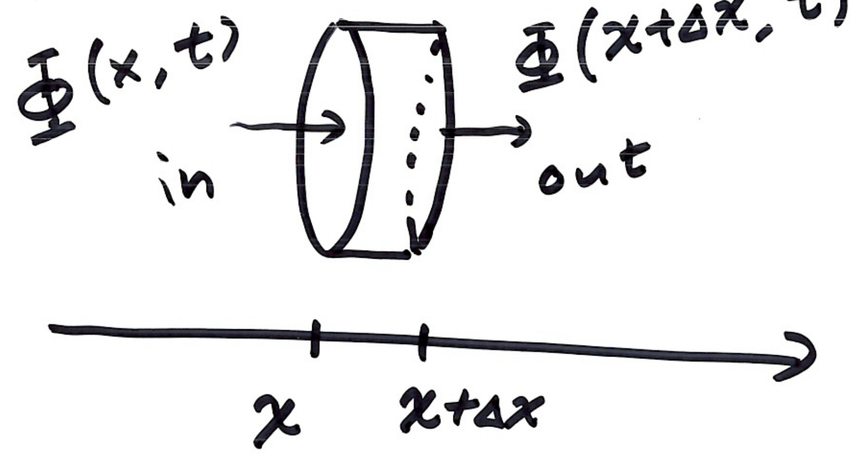
$u(x,t)$

Heat flux Φ: $\Phi(x,t)$

amount of heat
passing through a x-sec
per unit time
per unit area

$\Phi > 0$
if heat flows
in $+x$ direction

$\underline{\underline{\Phi}} \cdot \underline{\underline{\Delta t}} \cdot \underline{\underline{A}} =$ heat
(or energy)



Within Δt

in > out

$\Rightarrow$ net acc. of heat

$\quad$ ||

in minus out

$$\left[\Phi(x,t) - \Phi(x+\Delta x, t)\right] \cdot \Delta t \cdot \cancel{A}$$

$$= \underbrace{\Delta u}_{\text{change in temp.}} \cdot \underbrace{M}_{\text{density}} \cdot \underbrace{C_p}_{\text{specific heat}}$$

$$= \Delta u \cdot \cancel{A} \cdot \Delta x \cdot \rho \cdot C_p$$

M : mass

||

$\rho \cdot V$ — ρ: density, V: volume

||

$\rho \cdot A \cdot \Delta x$

$$\frac{\Delta u}{\Delta t} = \frac{-1}{\rho \cdot c_p} \cdot \frac{\Phi(x+\Delta x, t) - \Phi(x, t)}{\Delta x}$$

$\lim_{\substack{\Delta t \to 0 \\ \Delta x \to 0}}$

$$\boxed{\frac{\partial u}{\partial t} = \frac{-1}{\rho c_p} \frac{\partial \Phi}{\partial x}}$$

2 unknown vars

only 1 eq. !

Need another eq.

Fourier's law (empirical)

* heat flows from hot to cold spots
* |heat flux| $\propto$ |temperature gradient|

3-D $\cancel{\nabla T}$ ∇u

1-D $\frac{\partial u}{\partial x}$

$$\Phi \propto -\frac{\partial u}{\partial x}$$

$$\boxed{\Phi = -\kappa \frac{\partial u}{\partial x}}$$

$\kappa > 0$

$$\begin{cases} \dfrac{\partial u}{\partial t} = -\dfrac{1}{\rho c_p}\dfrac{\partial \Phi}{\partial x} & \text{— (1)} \\[2ex] \Phi = -\kappa \dfrac{\partial u}{\partial x} & \text{— (2)} \end{cases}$$

(1) + (2)

$$\frac{\partial u}{\partial t} = \left(\frac{\kappa}{\rho c_p}\right)\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\kappa}{\rho c_p} \equiv K > 0$$

$$\boxed{\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}}$$

1-D Heat eq.