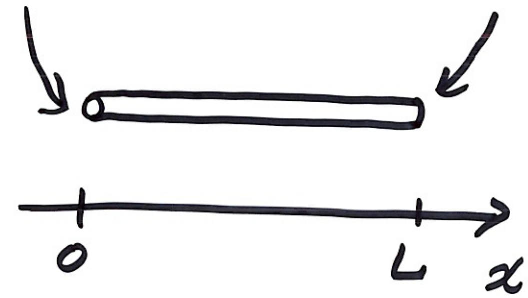
Lecture 4

Recap Lecture 3

Derived <u>1-D Heat</u> eq.



$u(x,t)$

* Local heat budget

Φ heat flux

$\triangleq$

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho c_p}\frac{\partial \Phi}{\partial x} \quad — ①$$

all $\rho,\ c_p,\ \kappa > 0$

* Fourier's law (empirical)

$$\Phi = -\kappa \frac{\partial u}{\partial x} \quad — ②$$

① + ② $$\boxed{\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2}}$$

$$K \equiv \frac{\kappa}{\rho c_p} > 0$$

thermal diffusivity

$$\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2} \qquad \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) \qquad K > 0 \quad \underline{\text{const}}$$

$\hat{t} \equiv Kt$

$$\frac{\partial u}{\partial \hat{t}} = \frac{\partial^2 u}{\partial x^2} \qquad u(\hat{t}, x)$$

$\hat{x} \equiv \frac{x}{\sqrt{K}}$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial \hat{x}^2}$$

$L \rightarrow 1$

Hw1 K, L, etc.

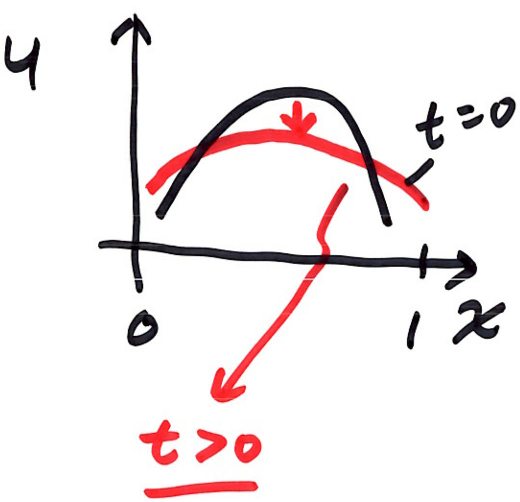
* suffices to solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

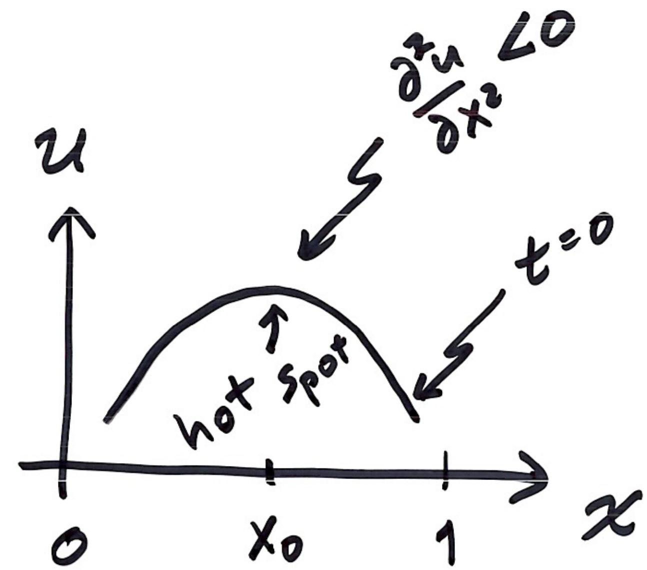
$0 \leq x \leq 1$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$\boxed{\frac{\partial u}{\partial t} \rightarrow \left(\frac{\partial u}{\partial t}\right)_x}$$

=





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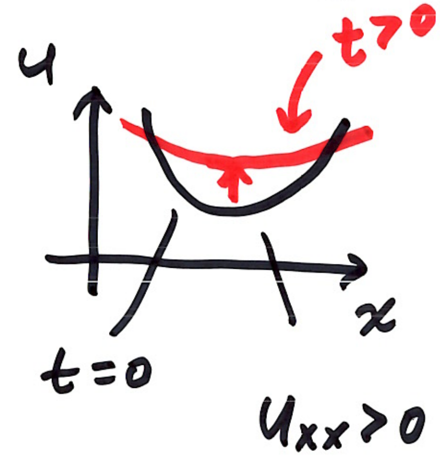
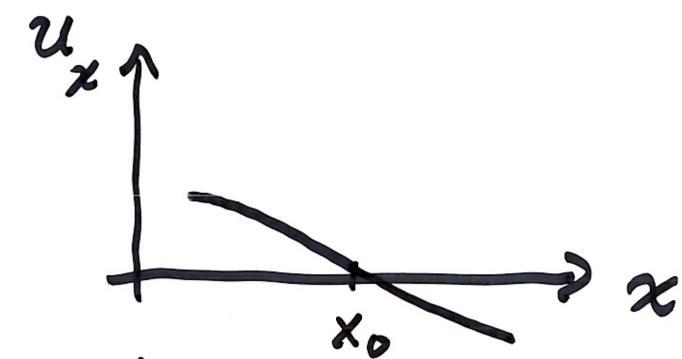
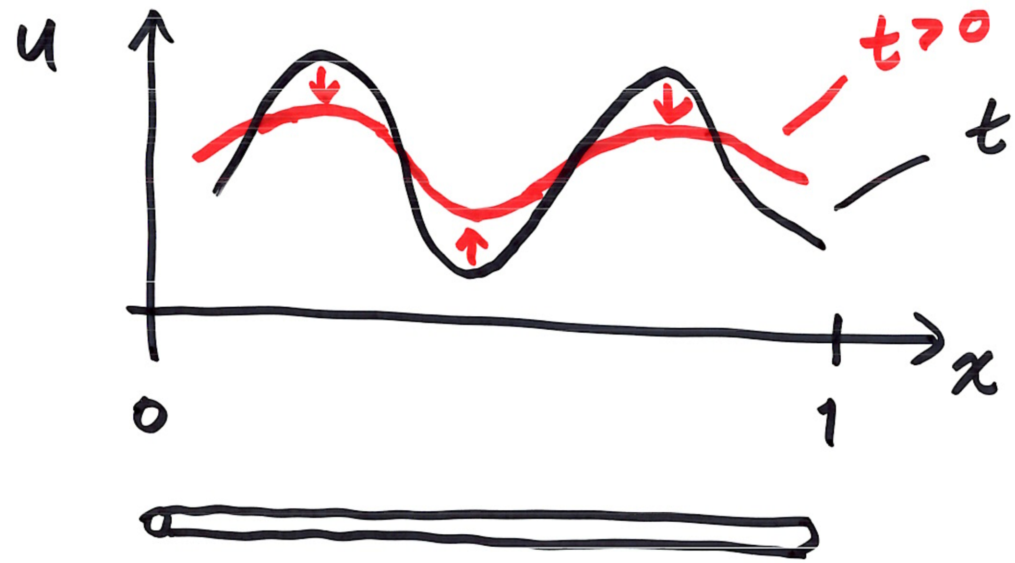






diffusive

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

as $t \to \infty$

"steady solution" $\longrightarrow$ u_s

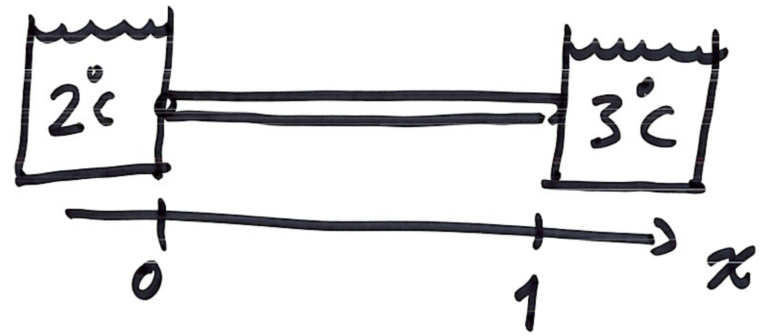
$u_s(x)$

$$\frac{\partial u_s}{\partial t} = 0$$

$$\Downarrow$$

$$\underline{u_s(x) = Ax + B} \Longleftarrow \frac{\partial^2 u_s}{\partial x^2} = 0$$

$u(x,t)$



$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} \\ u(0,t) = 2 \checkmark \\ u(1,t) = 3 \checkmark \\ u(x,0) = P(x) \leftarrow \text{given, initial dist. of temp.} \end{cases}$$

$u(x,t)$ sol.

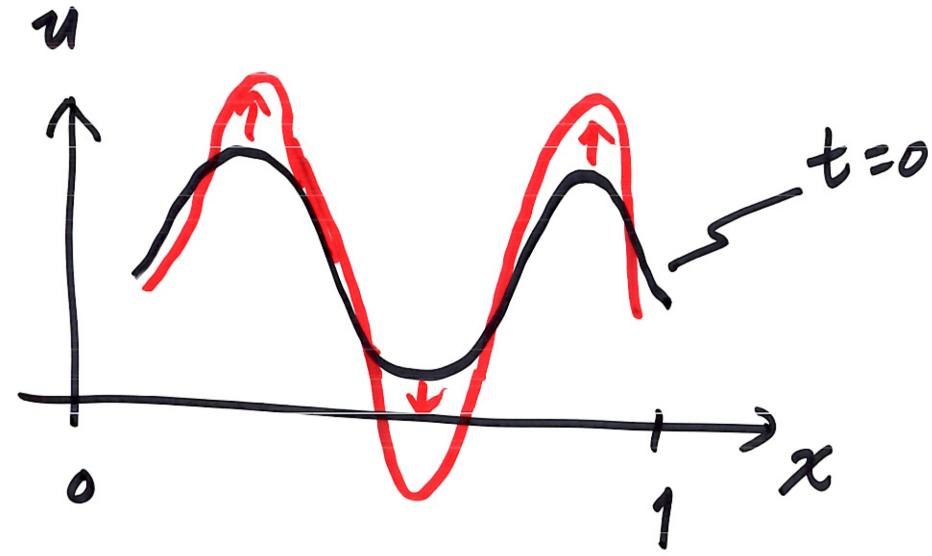
$u(x,t) = ?$ as $t \to \infty$

$$\begin{cases} \dfrac{d^2 u_s}{dx^2} = 0 \\ u_s(0) = 2 \\ u_s(1) = 3 \end{cases} \Rightarrow \underline{u_s(x) = Ax + B} \Rightarrow \boxed{u_s(x) = x + 2}$$

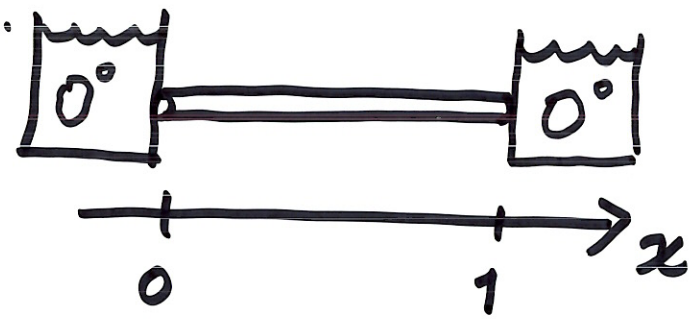
Some system $u(x,t)$

$$\frac{\partial u}{\partial t} = - \frac{\partial^2 u}{\partial x^2}$$

anti-diffusive



* End-to-end solution for 1-D Heat eq. (simplest case)



$u(x,t)$ $\quad 0 \le x \le 1 \quad t \ge 0$

$$\left\{\begin{array}{ll} & \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} \quad (*) \\ (\mathrm{I}) & u(0,t) = 0 \\ (\mathrm{II}) & u(1,t) = 0 \\ (\mathrm{III}) & u(x,0) = P(x) \end{array}\right.$$

given

first step ✓

"seperation of variables"

for constructing the b.b.'s (NOT (yet) finding the full sol.)

homogenous sub-system

$u(x,t) \sim G(x)H(t)$

understood as a candidate of b.b.

*NOT full solution !!!! *

Lecture 2

magic building blocks

$U(x)$

$$\frac{d^2 U}{dx^2} = c\,U$$

U $\quad c<0$

$\sin(\sqrt{-c}\,x)$

cos

sinh

cosh

$$u \sim G(x)H(t)$$

$$(*) \implies \frac{\partial(GH)}{\partial t} = \frac{\partial^2(GH)}{\partial x^2}$$

$$\frac{1}{GH} \qquad G\frac{dH}{dt} = H\frac{d^2G}{dx^2}$$

$$\underbrace{\frac{1}{H}\frac{dH}{dt}}_{\mathcal{H}(t)} = \underbrace{\frac{1}{G}\frac{d^2G}{dx^2}}_{\mathcal{G}(x)} = c$$

$$\mathcal{H}(t) = \mathcal{G}(x) = \underline{\text{const}}$$

"Common" constant

$$\begin{cases} \dfrac{d^2G}{dx^2} = cG \\ \dfrac{dH}{dt} = cH \end{cases}$$

$\dfrac{d}{dt}$ "d" <u>dee</u>

$\dfrac{\partial}{\partial x}$ "∂" <u>partial</u>

"∇" <u>del</u>

$$u \sim G(x)H(t)$$

b.c. (I) $\underline{u(0,t)=0}$ $\quad\Rightarrow\quad G(0)H(t)=0$

$$\Downarrow$$

$\underline{G(0)=0}$

either $\underline{H(t)=0}$ $\Rightarrow$ $u \sim G(x)H(t) = \underline{\underline{0}}$

building block = "NOTHING"

or $G(0)=0$

b.c. (II) $u(1,t)=0$

$\rightarrow\ G(1)H(t)=0$

$\Rightarrow\ \underline{G(1)=0}$

$$\boxed{\begin{array}{ll} \dfrac{d^2G}{dx^2} = cG & \begin{array}{l} G(0)=0 \\ G(1)=0 \end{array} \\[2ex] \dfrac{dH}{dt} = cH & \end{array}}$$

solve eigenvalue prob