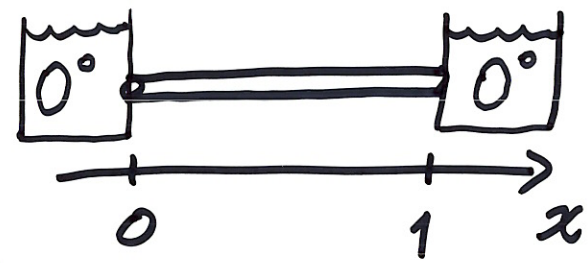
Lecture 5

End-to-end solution of 1-D Heat eq.

Recap (Lec 4)

$u(x,t)$ $\quad t \geq 0$, $\quad 0 \leq x \leq 1$



homogeneous sub-system

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{— } (*)$$

(I) $u(0,t) = 0$

(II) $u(1,t) = 0$

(III) $u(x,0) = p(x)$ ← given

"Separation of var"

$u(x,t) \sim G(x)H(t)$

↳ NOT yet the full sol., but merely a candidate of the b.b.

Sep. of var $\rightsquigarrow$ (*) $\Rightarrow$ $\left\{ \begin{array}{l} \boxed{\dfrac{d^2G}{dx^2} = c\,G \qquad G(0)=0^{\,(I)},\ G(1)=0^{\,(II)}} \\ \dfrac{dH}{dt} = c\,H \end{array} \right.$

$u \sim G(x)H(t)$

Sep of var (I) (II)

eigenvalue prob.

$$\boxed{(\ )' \equiv \frac{d(\ )}{dx}}$$

$$G'' = c\,G \qquad G(0) = 0 \ \text{(I)} \qquad G(1) = 0 \ \text{(II)} \qquad 0 \le x \le 1$$

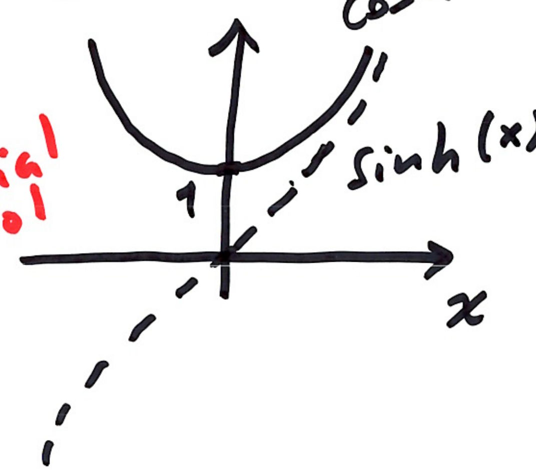
Lec 2/3?

(i) If $c > 0$

general sol: $G(x) = A\cosh(\sqrt{c}\,x) + B\sinh(\sqrt{c}\,x)$

b.c. (I): $0 = G(0) = A\underbrace{\cosh(0)}_{1} + B\cancel{\sinh(0)} \to 0$

$\Rightarrow \underline{A = 0}$



b.c. (II): $0 = G(1) = B\sinh(\sqrt{c})$

$\Rightarrow$ either $B = 0 \Rightarrow G(x) \equiv 0$ ✗ trivial sol

or $\sinh(\sqrt{c}) = 0$

$\Rightarrow c = 0$ contradiction ✗

(ii) If $c=0$ $\quad G''=0 \Rightarrow G(x) = A + Bx$

general sol.

b.c. (I) $\Rightarrow$ $0 = G(0) = A + B\cdot 0 = A$

$\Rightarrow \underline{A=0}$

b.c. (II) $\Rightarrow$ $0 = G(1) = B \cdot 1 \Rightarrow \underline{B=0}$

$\Rightarrow G(x) \equiv 0$ trivial

(iii) If $c<0$

general sol: $G(x) = A\cos(\sqrt{-c}\,x) + B\sin(\sqrt{-c}\,x)$

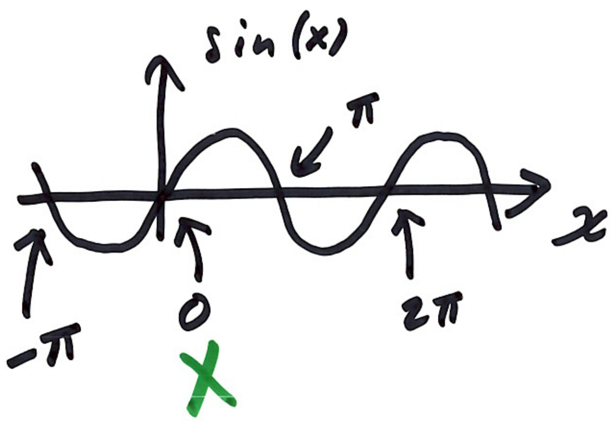
b.c. (I) : $0 = G(0) = A\underbrace{\cos(0)}_{1} + B\cancel{\sin(0)}\to 0$

$\Rightarrow \underline{A=0}$

b.c. (II) : $0 = G(1) = B\sin(\sqrt{-c})$

$\Longrightarrow$ either $\underline{B=0} \Rightarrow$ trivial ✗

or $\boxed{\sin(\sqrt{-c})=0}$



$\Rightarrow \sqrt{-c} = \pm\pi, \pm 2\pi, \pm 3\pi, \dots \pm n\pi, \dots$

$-c = \pi^2, 4\pi^2, 9\pi^2, \dots, (n\pi)^2, \dots$

$c = -\pi^2, -4\pi^2, -9\pi^2, \dots, -(n\pi)^2, \dots$

<u>eigenvalues</u>

<u>eigenfunctions</u> : $G(x) = \sin(\pi x), \sin(2\pi x), \sin(3\pi x), \dots$

We are not seeking the <u>full solution</u>
but only the <u>b.b.</u>

b.b. in
x direction :

$$C_n = -(n\pi)^2 \quad n = 1, 2, 3, \ldots$$

$$\underline{G_n(x) = \sin(n\pi x)}$$

$B \sin(n\pi x)$ $\quad n = 1, 2, 3, 4, \ldots$

we <u>don't</u> care what it is !

OK to set it to 1 (or 1 trillions if you prefer !)

-1

in t- direction (Note: $c_n = -(n\pi)^2$, $n=1,2,3,\ldots$ are now <u>known</u> and <u>fixed</u>)

$$\frac{dH_n}{dt} = c_n H_n = -(n\pi)^2 H_n$$

$$\Rightarrow \quad H_n(t) = \underbrace{H_n(0)}_{\text{don't care, OK to set to 1}} e^{-(n\pi)^2 t}$$

b.b. in t-dir: $H_n(t) = e^{-(n\pi)^2 t}$, $n=1,2,3,\ldots$

full$_{b.b.}$ $u \sim G(x)H(t)$

	G	H	$u \sim GH$
$c_1 = -\pi^2$	$G_1(x) = \sin(\pi x)$	$H_1(t) = e^{-\pi^2 t}$	$u_1 \sim \sin(\pi x)\, e^{-\pi^2 t}$
$c_2 = -4\pi^2$	$G_2(x) = \sin(2\pi x)$	$H_2(t) = e^{-4\pi^2 t}$	$u_2 \sim \sin(2\pi x)\, e^{-4\pi^2 t}$
	$\vdots$	$\vdots$	$\vdots$

~~Full~~ Full sol. $\underline{u(x,t)}$

$$u(x,t) = a_1\, u_1(x,t) + a_2\, u_2(x,t) + a_3\, u_3(x,t) + \dots$$

$$= \sum_{n=1}^{\infty} a_n\, u_n(x,t)$$

$$= \sum_{n=1}^{\infty} \underline{a_n} \sin(n\pi x)\, e^{-(n\pi)^2 t} \quad \text{—— } (\star)$$

The only remaining mystery

Next : Want to use b.c. (III) to fix $\{a_n\}$

$u(x,0) = \underline{P(x)}$ ← given

take (✱) and set "t" to 0 in order to match

$$\Rightarrow \boxed{u(x,0) = \sum_{n=1}^{\infty} \underline{a_n} \sin(n\pi x) = P(x)}$$

unknown we want to solve

given

invoke "<u>orthogonality relation</u>" for $\{\sin(n\pi x)\}$

n integer

$$\int_0^1 \sin(n\pi x)\sin(m\pi x)\,dx = \begin{cases} 0 & n \neq m \\ 1/2 & n = m \neq 0 \\ 0 & n = m = 0 \end{cases}$$

proof only involves Calculus 101

$$\sum_{n=1}^{\infty} a_n \underbrace{\int_0^1 \sin(n\pi x)\sin(N\pi x)\,dx}_{\begin{cases} 0 & \text{if } n \neq N \\ \frac{1}{2} & \text{if } n = N \neq 0 \end{cases}} = \int_0^1 P(x)\sin(N\pi x)\,dx$$

Say, we want $\underline{a_N}$

say a_7

$$\frac{1}{2} a_N = \int_0^1 P(x)\sin(N\pi x)\,dx$$

$$\underline{a_N = 2\int_0^1 P(x)\sin(N\pi x)\,dx}$$

for any integer N

obtain all $\{a_n\}$

$\Rightarrow$ Full solution #