Lecture 6

1/30

* HW 1 posted
* TA information
* Exam schedule

} updated on Canvas.

Recap (Lec 4, 5) End-to-end sol. of 1-D Heat eq.

$\neq$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad (*)$$

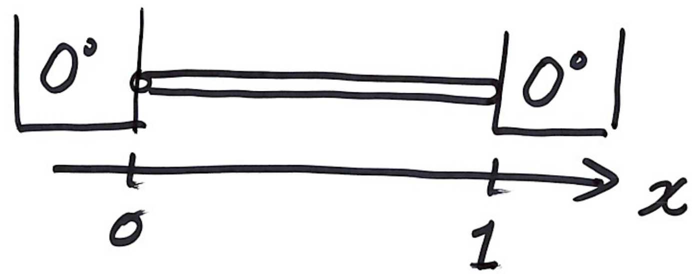
$$u(0,t) = 0 \quad (\mathrm{I})$$

$$u(1,t) = 0 \quad (\mathrm{II})$$

$$u(x,0) = P(x) \quad (\mathrm{III})$$

given

homogeneous subsystem



first half of the story

<u>sep. of var.</u> on (*), (I), (II)

$u \sim G(x)H(t) \leftarrow$ b.b., NOT full sol.

$$\frac{d^2G}{dx^2} = cG \qquad G(0)=0, \quad G(1)=0 \quad ☆$$

$$\frac{dH}{dt} = cH$$

$$c_n = -(n\pi)^2$$

$$H_n(t) = e^{-(n\pi)^2 t} \qquad G_n(x) = \sin(n\pi x), \quad n=1,2,3,\ldots$$

glue

full b.b. $u_n(x,t) = G_n(x)H_n(t) = \sin(n\pi x)\, e^{-(n\pi)^2 t}$

$n = 1, 2, 3, \ldots$

<u>full solution</u>:

$$u(x,t) = \sum_{n=1}^{\infty} a_n u_n(x,t) = \sum_{n=1}^{\infty} \underline{\underline{a_n}} \sin(n\pi x)\, e^{-(n\pi)^2 t} \quad (☆☆)$$

only unknowns left

second half of the story

<u>Set t to 0 in $(\ast\ast)$</u> to match with

b.c. (III)

$$p(x) = u(x,0) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)$$

Need to solve:

$$\boxed{\sum_{n=1}^{\infty} \underline{\underline{a_n}} \sin(n\pi x) = p(x)} \quad (\ast\ast\ast)$$

given

unknown

invoke "orthogonality relation" for $\{\sin(n\pi x)\}$

$$\Rightarrow \quad a_n = \boxed{}$$

plugging a_n to $(\ast\ast)$ $\Rightarrow$ Full sol. of $u(x,t)$

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Orthogonality relation

for $\{\sin(n\pi x)\}$ $\quad \begin{matrix} n \\ m \end{matrix}$: integer

$$\int_0^1 \sin(n\pi x)\sin(m\pi x)\,dx = \begin{cases} 0 & \text{if } n \neq m \\ 1/2 & \text{if } n = m \neq 0 \\ 0 & \text{if } n = m = 0 \end{cases}$$

for $\{\cos(n\pi x), 1\}$ $\quad \begin{matrix} n \\ m \end{matrix}$: integer

$$\int_0^1 \cos(n\pi x)\cos(m\pi x)\,dx = \begin{cases} 0 & \text{if } n \neq m \\ 1/2 & \text{if } n = m \neq 0 \\ \underline{\underline{1}} & \text{if } n = m = 0 \end{cases}$$

cf. HW1 Q1(a)

Proof case: $n \neq m$

trig identity

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$$

$$\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$$

$$I = \int_0^1 \sin(n\pi x)\sin(m\pi x)\,dx$$

$$J = \int_0^1 \cos(n\pi x)\cos(m\pi x)\,dx$$

$$I + J = \int_0^1 \cos((n-m)\pi x)\,dx = \frac{1}{(n-m)\pi}\left[\sin((n-m)\pi x)\Big|_{x=0}^{x=1}\right] = 0$$

$$J - I = \int_0^1 \cos((n+m)\pi x)\,dx = \frac{1}{(n+m)\pi}\left[\sin((n+m)\pi x)\Big|_{x=0}^{x=1}\right] = 0$$

H.S.A. $\begin{cases} J + I = 0 \\ J - I = 0 \end{cases} \Longrightarrow I = 0,\ J = 0$ #

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if $n = m \neq 0$

$$I = \int_0^1 \sin^2(n\pi x)\,dx = \dots = \frac{1}{2}$$

$$J = \int_0^1 \cos^2(n\pi x)\,dx = \dots = \frac{1}{2}$$

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if $n = m = 0$

$$I = \int_0^1 0\,dx = 0$$

$$J = \int_0^1 1\,dx = 1$$

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$$\int_0^1 dx \left(\sum_{n=1}^{\infty} a_n \sin(n\pi x) \sin(7\pi x) = P(x) \sin(7\pi x) \right)$$

say, we desire to compute $\underline{a_7}$

$$\Downarrow$$

$$\sum_{n=1}^{\infty} a_n \left[\int_0^1 \sin(n\pi x) \sin(7\pi x)\, dx \right]$$

$$= \int_0^1 P(x) \sin(7\pi x)\, dx$$

L.H.S.

$$= a_1 \cdot 0 + a_2 \cdot 0 + a_3 \cdot 0 + a_4 \cdot 0 + \ldots + \underline{a_7 \cdot \tfrac{1}{2}} + a_8 \cdot 0 + a_9 \cdot 0 + \ldots$$

$$\frac{1}{2} a_7 = \int_0^1 P(x) \sin(7\pi x)\, dx$$

$$\boxed{a_7 = 2 \int_0^1 P(x) \sin(7\pi x)\, dx}$$

in general,

$$\boxed{a_n = 2 \int_0^1 P(x) \sin(n\pi x)\, dx \quad n = 1, 2, 3, \ldots}$$

Second example

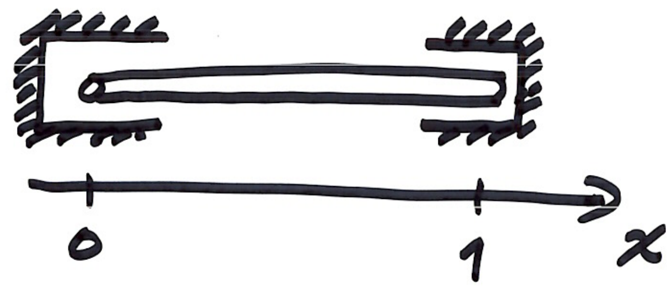
1-D Heat eq. + b.c. (I),(II) imposed on $\partial u/\partial x$

homog. sub-system

$$\left\{\begin{array}{ll} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} & \text{— } (*) \\ u_x(0,t)=0 & \text{— (I)} \\ u_x(1,t)=0 & \text{— (II)} \\ u(x,0)=P(x) & \text{— (III)} \end{array}\right.$$

sep. of var on (*)

$$\left\{\begin{array}{l} \dfrac{d^2G}{dx^2} = cG \\ \dfrac{dH}{dt} = cH \end{array}\right.$$



end-points "sealed off"

heat flux $\Phi = 0$ at $x = 0, 1$

Fourier's law

$$\Phi \propto -\frac{\partial u}{\partial x}$$

u_x denotes $\dfrac{\partial u}{\partial x}$

sep. of var to b.c. (I)

$$u \sim G(x)H(t) \qquad u_x \equiv \frac{\partial u}{\partial x} \equiv \frac{\partial (GH)}{\partial x} = H\frac{dG}{dx}$$

b.c. (I) $\Rightarrow$ $G'(0)\,H(t)=0$ $\qquad (\,)' \equiv \dfrac{d(\,)}{dx}$

$\Rightarrow$ either $H(t)=0$ $\times$ $\rightarrow$ trivial

or $G'(0)=0$

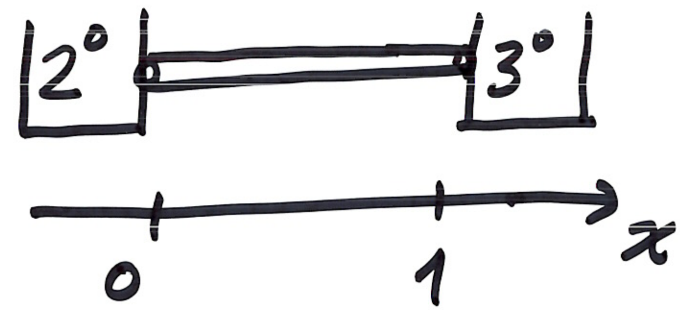
b.c. (II) $\Rightarrow \dots \Rightarrow G'(1)=0$

$$\Rightarrow \begin{cases} \boxed{\dfrac{d^2G}{dx^2} = cG \qquad G'(0)=0 \qquad G'(1)=0} \\ \dfrac{dH}{dt} = cH \end{cases}$$

to be continued in Lecture 7

Side note: non-homogeneous b.c.

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} \\ u(0,t) = 2 \quad (\mathrm{I}) \\ u(1,t) = 3 \quad (\mathrm{II}) \\ u(x,0) = p(x) \end{cases}$$



Non homog. b.c. !

Not separable !!

Say, take (I)

try sep. of var: $u \sim G(x)H(t)$

$$\Rightarrow \quad G(0)H(t) = 2 \quad \Rightarrow \quad G(0) = \frac{2}{H(t)}$$

b.c. (II) $\Rightarrow \quad G(1) = \dfrac{3}{H(t)}$

$$\text{“} \quad \frac{d^2G}{dx^2} = cG \qquad G(0) = \frac{2}{H(t)}, \quad G(1) = \frac{3}{H(t)} \quad \text{”}$$