

(Continue Lec 6) Second example of end-to-end solution

$$\left\{ \begin{array}{l} \left[\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \right] \quad (*) \\ u_x(0, t) = 0 \quad (I) \\ u_x(1, t) = 0 \quad (II) \\ u(x, 0) = P(x) \quad (III) \end{array} \right.$$

homogeneous sub-system

given

(Lec 6): sep of var.

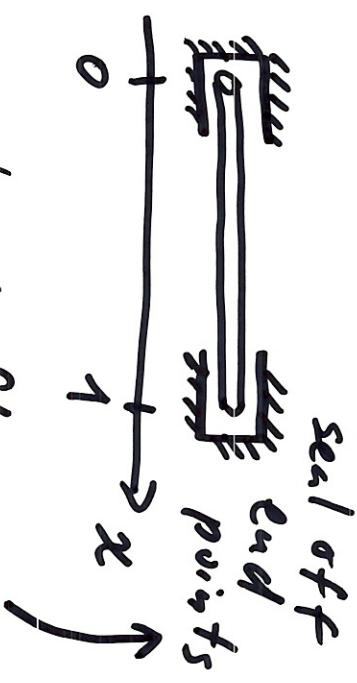
$$u \sim G(x)H(t)$$

$$()' \equiv \frac{d}{dx}$$

$$\frac{d^2 G}{dx^2} = cG \quad G'(0) = 0 \quad G'(1) = 0$$

$$\frac{dH}{dt} = cH$$

(*)



heat flux = 0

$$\Phi \propto -\frac{\partial u}{\partial x}$$

math b.c.:

$$\frac{\partial u}{\partial x} \leftarrow \begin{array}{l} u_x(0, t) = 0 \\ u_x(1, t) = 0 \end{array}$$

$$G'' = cG \quad G'(0) = 0 \quad G'(1) = 0$$

(i) (ii)

If $c > 0$

$$G(x) = A \cosh(\sqrt{c}x) + B \sinh(\sqrt{c}x)$$

$$\Rightarrow G'(x) = \sqrt{c} A \sinh(\sqrt{c}x) + \sqrt{c} B \cosh(\sqrt{c}x)$$

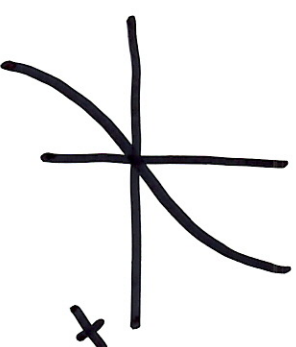
$$\text{b.c. (i)} \quad 0 = G'(0) = \sqrt{c} A \underbrace{\sinh(0)}_0 + \sqrt{c} B \underbrace{\cosh(0)}_1$$

$$\Rightarrow \sqrt{c} B = 0 \Rightarrow \underline{B = 0}$$

$$\text{b.c. (ii)} \quad 0 = G'(1) = \sqrt{c} A \sinh(\sqrt{c})$$

$$\Rightarrow \text{either } A = 0 \quad \text{or } \sinh(\sqrt{c}) = 0 \Rightarrow c = 0$$

$$\Rightarrow \text{trivial} \quad \text{or } \text{contradiction}$$



$$\text{If } c=0 \Rightarrow G''=0$$

$$\Rightarrow G(x) = A + Bx$$

$$\Rightarrow G'(x) = B$$

$$\text{b.c. (i)} \Rightarrow 0 = G'(0) = B \Rightarrow B = 0$$

$$\text{b.c. (ii)} \Rightarrow 0 = G'(1) = B \Rightarrow B = 0$$

A remains undetermined

A can be of any values!

* $c=0$ is an eigenvalue

\Rightarrow eigenfunction is $G(x) = \text{constant}$ 

$G_0(x)$ "plateau"



$$G_0(x) = 1$$

OK to set to 1

is a b.b.

If $c < 0$

$$G(x) = A \cos(\sqrt{-c} x) + B \sin(\sqrt{-c} x)$$

$$\Rightarrow G'(x) = -\sqrt{-c} A \sin(\sqrt{-c} x) + \sqrt{-c} B \cos(\sqrt{-c} x)$$

$$\text{b.c. (i)} \Rightarrow 0 = G'(0) = -\sqrt{-c} A \underbrace{\sin(0)}_0 + \sqrt{-c} B \underbrace{\cos(0)}_1$$

$$\Rightarrow B = 0$$

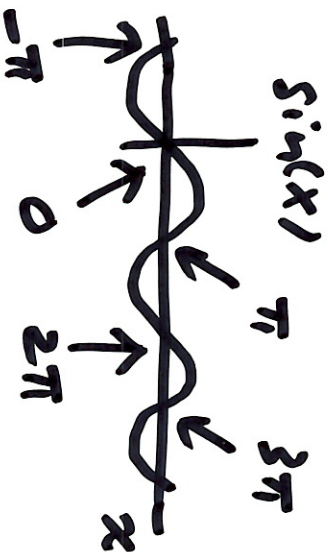
$$\text{b.c. (ii)} \Rightarrow 0 = G'(1) = -\sqrt{-c} A \sin(\sqrt{-c})$$

same as

$$\Rightarrow \text{either } A = 0 \text{ or } \sin(\sqrt{-c}) = 0$$

$$\text{or } \sin(\sqrt{-c}) = 0$$

"First" Example



$$\Rightarrow \sqrt{-c} = \pm \pi, \pm 2\pi, \dots$$

$$c = -\pi^2, -4\pi^2, \dots$$

Loc in 4, 5

Don't care: OK to set to 1

Eigenfunc.:

$$C_n = -(n\pi)^2, \quad n=1, 2, 3, \dots$$

$$G_n(x) = A \cos(n\pi x), \quad n=1, 2, 3, \dots$$

Next, solve $dH/dt = cH$ for $c=0$, $H = e^{-(n\pi)^2 t}$
 $n=1, 2, 3, \dots$

$$c=0 \Rightarrow \frac{dH_0}{dt} = 0 \Rightarrow H_0(t) = \underline{\text{const.}}$$

Don't care, OK to
set to 1

$$\underline{H_0(t) = 1}$$

$$c \neq 0 \Rightarrow$$

$$\frac{dH_n}{dt} = c_n H_n = -(n\pi)^2 H_n \quad n=1, 2, \dots$$

$$H_n(t) = \underline{H_n(0) e^{-(n\pi)^2 t}}$$

Don't care, OK to
set to 1

give $G(x)$ and $H(t)$

c	G_n	H_n	$u_n = G_n H_n$
$c = 0$	$G_0(x) = 1$	$H_0(t) = 1$	$u_0(x, t) = 1$
$c = -\pi^2$	$G_1(x) = \cos(\pi x)$	$H_1(t) = e^{-\pi^2 t}$	$u_1(x, t) = \cos(\pi x) e^{-\pi^2 t}$
$c = -4\pi^2$	$G_2(x) = \cos(2\pi x)$	$H_2(t) = e^{-4\pi^2 t}$	\vdots

* Full sol.

$$\begin{aligned}
 u(x, t) &= a_0 u_0(x, t) + a_1 u_1(x, t) + a_2 u_2(x, t) + \dots \\
 &= a_0 + \sum_{n=1}^{\infty} a_n u_n(x, t) \\
 &= a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) e^{-(n\pi)^2 t} \quad \text{---} (\star \star)
 \end{aligned}$$

Next step: Determine $\{a_n\}_{n=0,1,2,\dots}$
 from b.c. (III) $\underline{u(x,0) = p(x)}$

Set t to 0 in ~~(I)~~ to match \int given

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) = p(x) \quad (**)$$

n, m integers Recall:
 Orthogonality relation

$$\int_0^1 \cos(n\pi x) \cos(m\pi x) dx = \begin{cases} 0 & \text{if } n \neq m \\ 1/2 & \text{if } n = m \neq 0 \\ 1 & \text{if } n = m = 0 \end{cases}$$

\star (See Lec 6 for proof)

Say ~~we~~ we desire to compute a_N with $N \neq 0$

$$= \frac{1}{N\pi} \sin(N\pi x) \Big|_{x=0}^{x=1}$$

$$\int_0^1 (**) \cdot \cos(N\pi x) dx$$

$$a_0 \int_0^1 \cos(N\pi x) dx +$$

$$\sum_{n=1}^{\infty}$$

$$a_n \int_0^1 \cos(n\pi x)$$

$$\cos(N\pi x) dx = \int_0^1 p(x) \cos(N\pi x) dx$$

$$= \int_0^1 p(x) \cos(N\pi x) dx$$

0

by calc. 101

$$= \begin{cases} 0 & \text{if } n \neq N \\ \frac{1}{2} & \text{if } n = N \end{cases}$$

$$\frac{1}{2} a_N = \int_0^1 p(x) \cos(N\pi x) dx$$

Good

for $N=1, 2, 3, \dots$

$$a_N = 2 \int_0^1 p(x) \cos(N\pi x) dx$$

for $n=0$ i.e. to compute a_0

$$\int_0^1 1 \text{ (**) } dx$$

$$\Rightarrow \int_0^1 a_0 dx + \sum_{n=1}^{\infty} a_n \int_0^1 \cos(n\pi x) dx = \int_0^1 p(x) dx$$

$a_0 + 0 = \int_0^1 p(x) dx$

$$a_0 = \int_0^1 p(x) dx$$

all $\{a_n\}$ determined

#

Lec 4.5, 6.7
Ex 1 Ex 2 \rightarrow HW 1
 Q 1(a)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Ex 1

$$u(0, t) = 0$$

$$u(1, t) = 0$$

Ex 2

$$u_x(0, t) = 0$$

$$u_x(1, t) = 0$$

$$u(x, 0) = P(x)$$

Ex 1

$$u(x, t) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) e^{-(n\pi)^2 t}$$

$$a_n = 2 \int_0^1 P(x) \sin(n\pi x) dx$$

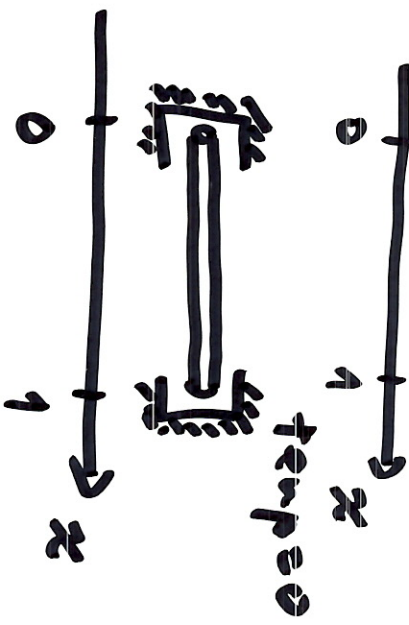
$$\text{s.s. } (t \rightarrow \infty)$$

$$u(x, \infty) \rightarrow \underline{0}$$

Ex 1



Ex 2



Ex 2

$$u(x, t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) e^{-(n\pi)^2 t}$$

$$n \neq 0$$

$$a_n = 2 \int_0^1 P(x) \cos(n\pi x) dx$$

$$a_0 = \int_0^1 P(x) dx$$

$$\text{s.s. } (t \rightarrow \infty)$$

$$u(x, \infty) \rightarrow \underline{a_0}$$

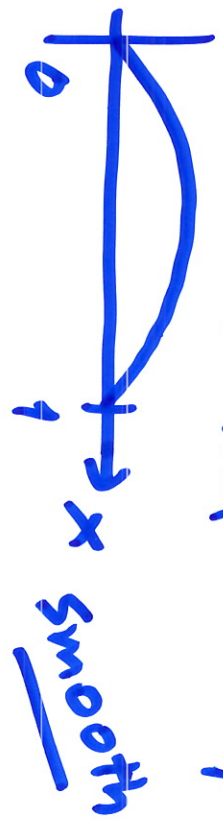
$$\sin(n\pi x) e^{-(n\pi)^2 t}$$

$$\sin(\pi x)$$

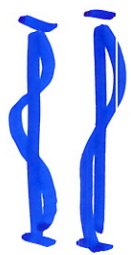
$n=1$

Exponential

decay



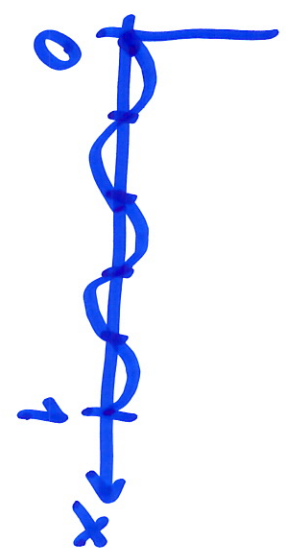
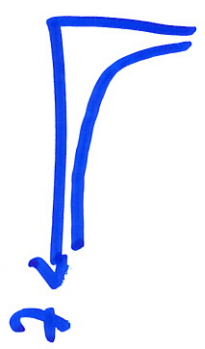
$$\sin(5\pi x)$$



$n=1$
 $n=5$

$$e^{-\pi^2 t}$$

$$e^{-25\pi^2 t}$$



in space

diffusive ✓

also for

$$\cos(n\pi x) e^{-(n\pi)^2 t}$$

$$\frac{E_{x^2}}{2}$$

Total energy $E(t) \equiv \int_0^1 u(x,t) dx$

$$\frac{dE}{dt} = ?$$

$$u(x,t)$$

$$\frac{dE}{dt} = \frac{d}{dt} \int_0^1 u dx$$

$$= \int_0^1 \frac{\partial u}{\partial t} dx$$

$$= \int_0^1 \frac{\partial^2 u}{\partial x^2} dx$$

$$= u_x(1,t) - u_x(0,t)$$

$$= 0 - 0$$

$$= 0$$

$$\Rightarrow E(t) = \text{const}$$

$$E(t) = E(0) = \underline{E(\infty)}$$

$$\underline{Q_0} = \int_0^1 p(x) dx = \int_0^1 u(x,0) dx$$