Lecture 9

Tips on Hw1 <u>Q2, Q3</u> (Q5)

Ex: <u>Q2</u>

$u(x,t)$

$$\frac{\partial u}{\partial t} = \left(\frac{2}{1+t}\right)\frac{\partial^2 u}{\partial x^2} + \left(\frac{8}{1+t}\right)u \quad *$$

<u>Sep of var?</u>

$u \sim \underline{G(x)\,H(t)}$

$$\frac{\partial (GH)}{\partial t} = \left(\frac{2}{1+t}\right)\frac{\partial^2 (GH)}{\partial x^2} + \left(\frac{8}{1+t}\right)GH$$

∂ | d

$$G\frac{dH}{dt} = \left(\frac{2}{1+t}\right)\cdot H\frac{d^2G}{dx^2} + \left(\frac{8}{1+t}\right)GH \qquad \frac{1}{GH}$$

$$\frac{1}{H}\frac{dH}{dt} = \left(\frac{2}{1+t}\right)\frac{1}{G}\frac{d^2G}{dx^2} + \left(\frac{8}{1+t}\right) \qquad \times\ (1+t)$$

$$\frac{1+t}{H}\frac{dH}{dt} = \frac{2}{G}\frac{d^2G}{dx^2} + 8$$

$$\underbrace{\frac{1+t}{H}\frac{dH}{dt}}_{H(t)} = \underbrace{\frac{2}{G}\frac{d^2G}{dx^2}+8}_{G(x)} = c$$

$$\begin{cases} (1+t)\dfrac{dH}{dt} = cH & \checkmark \\ \boxed{2\dfrac{d^2G}{dx^2} + \cancel{4}^{\,8}\, G = cG \quad \checkmark \qquad (+\ 2\ \text{b.c.'s for } G)} \end{cases}$$

OK

solve eig. v. prob.

Will work, but...

alternative

$$\frac{1+t}{H}\frac{dH}{dt} = \frac{2}{G}\frac{d^2G}{dx^2} + 8$$

$$\underbrace{\left(\frac{1+t}{2}\right)\frac{1}{H}\frac{dH}{dt} - 4}_{H(t)} = \underbrace{\frac{1}{G}\frac{d^2G}{dx^2}}_{G(x)} = c$$

$$\left\{\begin{array}{l} \boxed{\dfrac{d^2G}{dx^2} = cG \quad + \quad 2 \text{ b.c.'s for } G} \\ \left(\dfrac{1+t}{2}\right)\dfrac{1}{H}\dfrac{dH}{dt} = c+4 \end{array}\right.$$

easier to solve (we already solved it in class!)

will work

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same idea for Q3

Q5 :

physical (dimensional)

(1) $\left\{\begin{array}{l} \dfrac{\partial u}{\partial \hat{t}} = K \dfrac{\partial^2 u}{\partial \hat{x}^2} \\ 0 \le \hat{x} \le L \end{array}\right.$

our math eq.

(2) $\left\{\begin{array}{l} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} \\ 0 \le x \le 1 \end{array}\right.$

(u temp.)

K thermal diffusivity

L length of metal rod

$\hat{x} = L\,x$

$\hat{t} = T\,t$

(L, T, K)

$\hat{x}$: dim (m)

$\hat{t}$: dim (s)

x : non-dim

t : non-dim

$\boxed{\text{claim: (1) } \equiv \text{ (2) equivalent}}$

$\Leftrightarrow$ implies a relation among (L, T, K)

~~*~~ * 2D Heat eq.

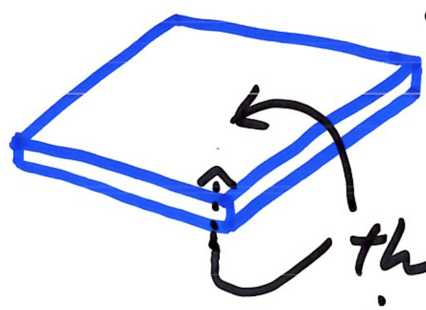
(3D)

✓ $\boxed{\text{* Laplace's eq.}}$

2D Heat eq.

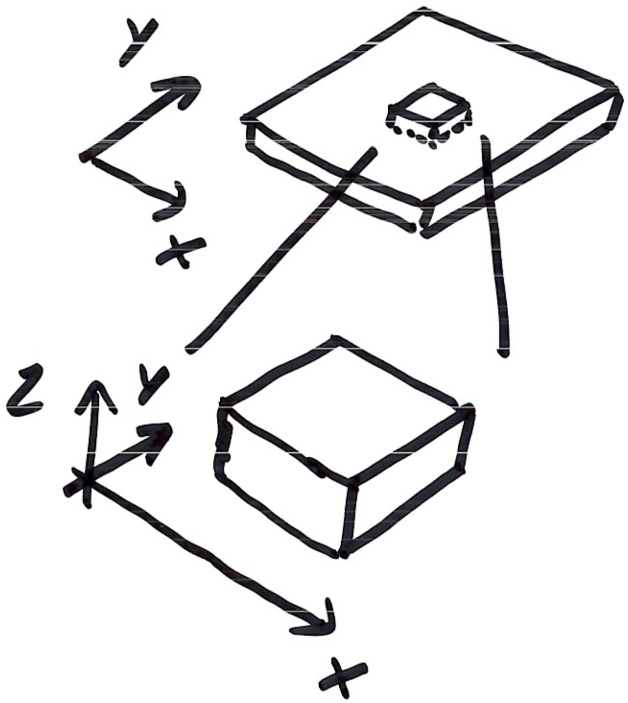
thin metal plate

1D Heat



$u(x, y, t)$

thermally insulate top/bottom boundary



vector heat flux (2D)

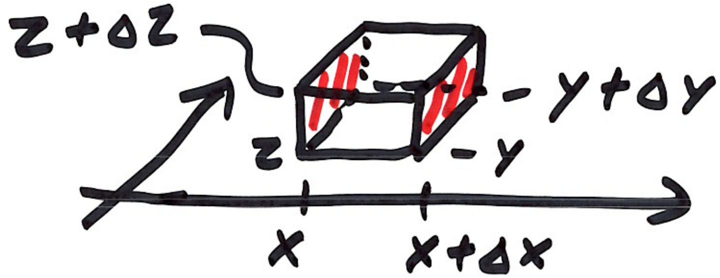
$$\vec{\Phi} = \phi_x \hat{i} + \phi_y \hat{j}$$

energy (heat) passing through a x-sec per unit <u>area</u> per unit <u>time</u>

First, restrict heat flow to x-direction

$$\vec{\Phi} = \phi_x \hat{i}$$

→ <u>story is identical to 1-D case</u>



$$\cancel{\Phi(x,z,}\left[\Phi_x(x,y,z,t) - \Phi_x(x+\Delta x, y, z, t)\right]\underline{\underline{\Delta y \Delta z}} \cdot \underline{\Delta t}$$

$$= \cancel{\rho}\, c_p \cdot \underbrace{\rho \cdot \Delta x \underline{\Delta y \Delta z}}_{\text{mass}} \cdot \Delta u$$

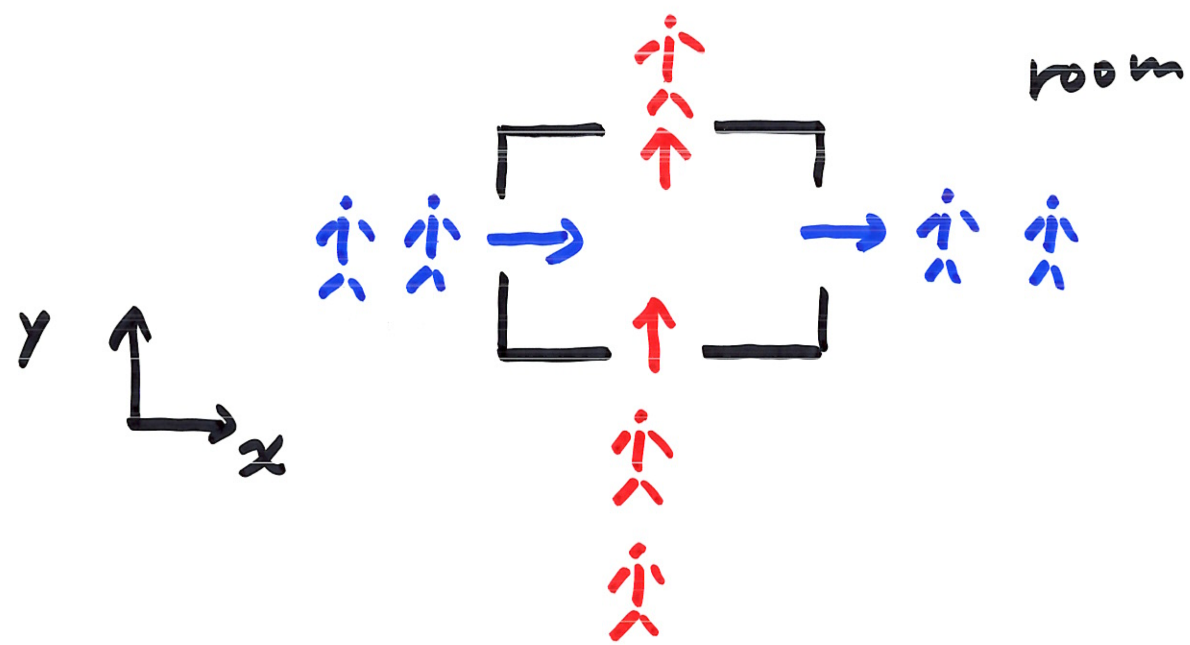
$$\lim_{\substack{\Delta t \to 0 \\ \Delta x \to 0}} \qquad \cancel{\frac{\partial \Phi_x}{\partial}} \qquad \underline{\frac{\partial u}{\partial t} = -\frac{1}{\rho c_p} \frac{\partial \Phi_x}{\partial x}} \quad (1)$$

Repeat, but restrict heat flow to y-direction $\vec{\Phi} = \phi_y \hat{j}$

$$\Rightarrow \quad \underline{\frac{\partial u}{\partial t} = -\frac{1}{\rho c_p} \frac{\partial \Phi_y}{\partial y}} \quad (2)$$

in general

$$\vec{\Phi} = \phi_x \hat{i} + \phi_y \hat{j} \qquad \frac{\partial u}{\partial t} = (1) + (2)$$


room
y
x

local heat budget

$$(*) \quad \frac{\partial u}{\partial t} = -\frac{1}{\rho c_p}\left(\frac{\partial \Phi_x}{\partial x} + \frac{\partial \Phi_y}{\partial y}\right)$$

f

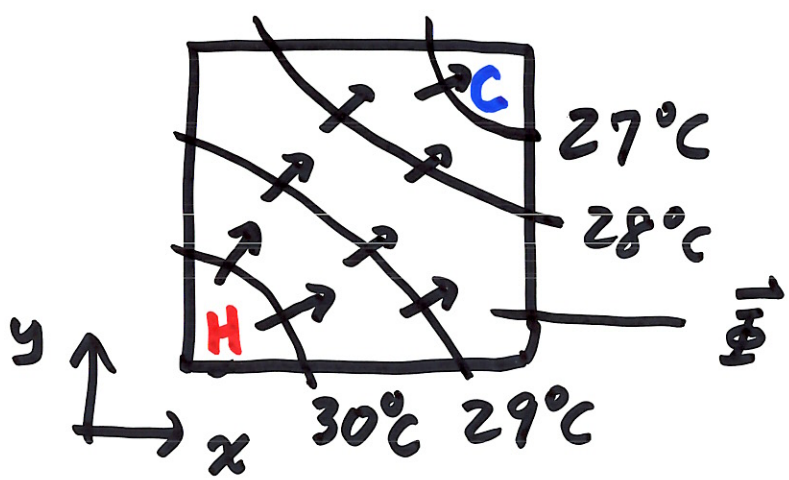
$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \dots$$

Need more eqs.

Fourier's law (2D)

* heat flows from hot to cold spots

* |heat flux| $\propto$ |temperature gradient|



$$\vec{\Phi} \propto -\nabla u$$

$$\vec{\Phi} = -\kappa \nabla u$$

$$\left.\begin{aligned} \Phi_x &= -\kappa \frac{\partial u}{\partial x} \\ \Phi_y &= -\kappa \frac{\partial u}{\partial y} \end{aligned}\right\} \checkmark \quad (**)$$

combine (*), (**)

$$\frac{\partial u}{\partial t} = \left(\frac{\kappa}{\rho c_p}\right)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\kappa}{\rho c_p} \equiv K > 0$$

$$\frac{\partial u}{\partial t} = K\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

2-D Heat eq.

3-D skip detail

$$\frac{\partial u}{\partial t} = K\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

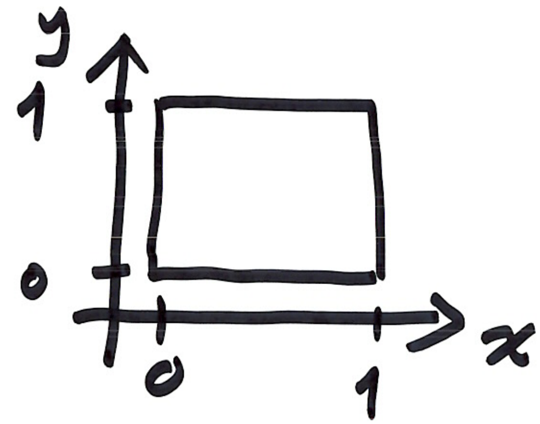
As before, let's ignore K ...

(non-dim)
2-D Heat.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

5 b.c.'s

2 in x-dir
2 in y-dir
1 in t-dir



Steady solution ? — independent of t

$u_s(x, y)$

Full sol. $u(x, y, t)$

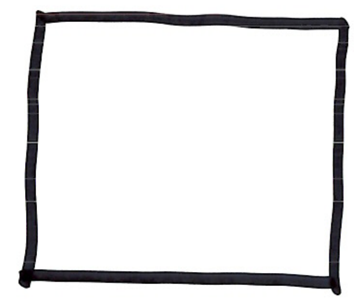
Laplace's eq.

$$\boxed{0 = \frac{\partial^2 u_s}{\partial x^2} + \frac{\partial^2 u_s}{\partial y^2} \quad + \text{ first 4 b.c.'s}}$$

Let's focus on Laplace's eq.

$u(x, y)$

$$\left[\begin{array}{l} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \\ + \ 4 \text{ b.c.'s} \end{array}\right.$$



it govern's the steady sol.
for the corresponding 2-D Heat eq.

1-D Heat eq $\longrightarrow$ steady sol. $u_s(x)$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad 0 = \frac{d^2 u_s}{dx^2}$$

$u_s(x)$ straight line

2-D Heat eq.

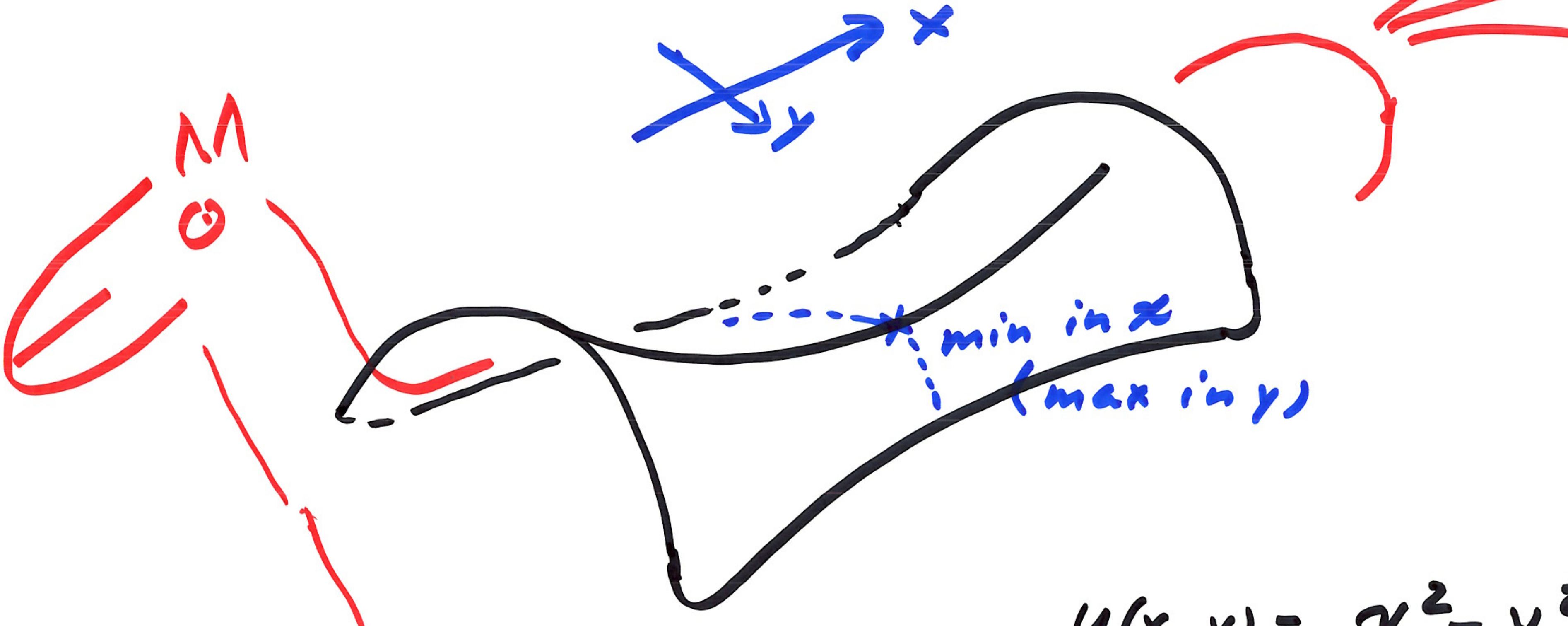
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \longrightarrow 0 = \frac{\partial^2 u_s}{\partial x^2} + \frac{\partial^2 u_s}{\partial y^2}$$

$u_s(x,y)$

$$\frac{\partial^2 u_s}{\partial x^2} = -\frac{\partial^2 u_s}{\partial y^2}$$

✓ plane

Saddle point



$$u(x,y) = x^2 - y^2$$

$(0,0)$